

DEPTH-ZERO BASE CHANGE FOR UNRAMIFIED $U(2, 1)$

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ABSTRACT. We give an explicit description of L -packets and quadratic base change for depth-zero representations of unramified unitary groups in two and three variables. We show that this base change is compatible with unrefined minimal K -types.

1. INTRODUCTION

Given a finite Galois extension E/F of finite, local, or global fields and a reductive algebraic F -group \underline{G} , “base change” is, roughly, a (sometimes only conjectural) mapping from representations of $G = \underline{G}(F)$ to those of $\underline{G}(E)$. When F is finite, or when F is local and $\underline{G} = \mathrm{GL}(n)$, then this mapping is the Shintani lifting (as introduced in [25] and extended in [17], [16], and [13] for finite groups).

Correspondences like base change that are associated to the Langlands program can be difficult to describe explicitly, even in cases where they are known to exist. Bushnell and Henniart [5, 6, 8, 7] are remedying this situation for base change for $\mathrm{GL}(n)$ over local fields. Analogously, Silberger and Zink [26] have made the Abstract Matching Theorem [11, 22, 2] explicit for depth-zero discrete series representations.

Suppose that F is a p -adic field of odd residue characteristic. If E/F is quadratic, and \underline{G} is a unitary group in three variables defined with respect to E/F , then Rogawski [23] has shown that a base change lifting exists, and has derived some of its properties. Our goal in this paper is to describe base change explicitly for depth-zero representations in the case where E/F is unramified. Depth-zero base change is particularly interesting because it should be closely related to base change for finite groups. See [19] for an exploration of another special case of this phenomenon.

In order to apply a technical lemma (Cor. 2.6), we will assume that the order q of the residue field k_F of F is at least 59. From the lemma, character identities can be verified by evaluation at “very regular” elements. At such elements, character values are particularly easy to compute. Without the lemma, the verification of these identities involves evaluation at more general elements. Character values at these elements can be computed, but are far more complicated.

Note that we assume that F has characteristic zero only so that we can apply results of Rogawski [23]. Our calculations apply equally well if F is a function field of odd residue characteristic.

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Conjecturally, one should be able to determine the depth of the representations in an L -packet from the associated Langlands parameter. Thus, liftings that arise from the Langlands correspondence should preserve depth, if depth is normalized correctly. In particular, depth-zero representations should go to depth-zero representations. We assume this throughout for base change and for endoscopic lifting from $U(1, 1) \times U(1)$ to $U(2, 1)$.

In §2, we present our notation, review the general notion of Shintani lifting, describe how it applies to the representations of certain finite reductive subquotients of G , and list all of the representations of G of depth zero. In §4, we give an explicit description of the depth-zero L -packets and A -packets for G . In §5, we determine the base change lift of each of these packets. In §6 we examine the relationship between base change and K -types, as defined by Bushnell-Kutzko [9] and as described by Moy-Prasad [21] or Morris [20]. Recall that a (minimal) K -type (or simply a “type”) of depth zero is a pair (G_x, σ) , where G_x is a parahoric subgroup of G , and σ is the inflation to G_x of an irreducible cuspidal representation of the finite reductive quotient \mathbf{G}_x of G_x . Since all of the data in this definition can be lifted in a natural way to similar data for $\tilde{G} = \underline{G}(E)$, we have a natural notion of base change for depth-zero types. Under the above assumption on the residue characteristic of the p -adic field F , we show that base change for depth-zero types is compatible with base change for representations (actually, A -packets of representations):

Theorem 1.1. *Suppose Π is a depth-zero A -packet for G , let $\tilde{\pi}$ denote the base-change lift of Π , and let $\pi \in \Pi$. Suppose $(G_x, \text{infl}(\sigma))$ is a type contained in π . Then $\tilde{\pi}$ contains $(\tilde{G}_x, \text{infl}(\tilde{\sigma}))$, where $\tilde{\sigma}$ is the base-change lift of σ from \mathbf{G}_x to $\tilde{\mathbf{G}}_x$.*

Note that the pair $(\tilde{G}_x, \text{infl}(\tilde{\sigma}))$ contains a type upon restriction to some parahoric subgroup of \tilde{G}_x . Thus, either it is itself a type, or it carries more information than a type.

In §7, we state a formula for the character of an induced representation. The formula itself is not new, but we need to assert that it holds for representations of groups that are not necessarily connected.

In order to describe explicit base change for all representations of $U(2, 1)$ (not just of depth zero), one needs to understand depth-zero base change not just for $U(2, 1)$ but for unitary groups in two variables as well. We deal with this briefly in §3.

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2. PRELIMINARIES

2.1. General notation and facts. For any nonarchimedean local field F , let \mathcal{O}_F denote its ring of integers, \mathfrak{p}_F the prime ideal in \mathcal{O}_F , and $k_F = \mathcal{O}_F/\mathfrak{p}_F$ the residue field. For any abelian extension E/F , let $\omega_{E/F}$ denote the character of F^\times arising via local class field theory.

We will use underlined letters to denote algebraic groups and will drop the underlining to indicate the corresponding groups of rational points. Given an algebraic F -group \underline{G} and a finite extension E/F , let $\tilde{\underline{G}} = R_{E/F}(\underline{G})$, where $R_{E/F}$ denotes

A-packet*	Base change lift
$\{\text{Ind}_B^G \lambda\}$ ($\text{Ind}_B^G \lambda$ irreducible and $\text{Ind}_B^G \tilde{\lambda}$ irreducible) (§4.1)	$\text{Ind}_B^G \tilde{\lambda}$ (§5.1)
$\{\text{Ind}_B^G \lambda\}$ ($\text{Ind}_B^G \lambda$ irreducible and $\text{Ind}_B^G \tilde{\lambda}$ reducible) (§4.1)	$\text{Ind}_P^G \left((\lambda_1 \tilde{\lambda}_2 \cdot _E^{\mp 1/2} \circ \det_{\text{GL}(2)}) \otimes \tilde{\lambda}_2 \right)$ (§5.1)
$\{\psi\}$ (one-dimensional) (§4.1)	$\tilde{\psi}$ (§5.1)
$\{\text{St}_G(\psi)\}$ (§4.1)	$\text{St}_{\tilde{G}}(\tilde{\psi})$ (§5.1)
$\{\pi_1(\lambda), \pi_2(\lambda)\}$ ($\text{Ind}_B^G \lambda = \pi_1(\lambda) \oplus \pi_2(\lambda)$) (§4.1)	$\text{Ind}_B^G \tilde{\lambda}$ (§5.1)
$\text{Ind}_{G_y}^G \sigma$, σ cubic cuspidal (§4.2)	$\text{Ind}_{\tilde{Z}\tilde{G}_y}^G \tilde{\sigma}$ (§5.2)
$\{\pi^2(\lambda), \pi^s(\lambda)\}$ (π^s described in Prop. 4.4) (§4.1)	$\text{Ind}_P^G \text{St}_{\tilde{H}} \left((\lambda_1 \tilde{\lambda}_2 \cdot _E^{\mp 1/2} \circ \det_{\text{GL}(2)}) \otimes \tilde{\lambda}_2 \right)$ (Prop. 5.6)
$\{\pi^n(\lambda), \pi^s(\lambda)^{(*)}\}$ (π^s described in Prop. 4.4) (§4.1)	$\text{Ind}_P^G \left((\lambda_1 \tilde{\lambda}_2 \cdot _E^{\mp 1/2} \circ \det_{\text{GL}(2)}) \otimes \tilde{\lambda}_2 \right)$ (Prop. 5.6)
$\left\{ \begin{array}{l} \text{Ind}_{G_y}^G \text{infl}_{G_y}^{G_y} (-R_{\tilde{C}^y}^{G_y} \varphi_1 \otimes \varphi_2 \otimes \varphi_3), \\ \text{Ind}_{G_z}^G \text{infl}_{G_z}^{G_z} (-R_{\tilde{C}^z}^{G_z} \varphi_1 \otimes \varphi_2 \otimes \varphi_3), \\ \text{Ind}_{G_z}^G \text{infl}_{G_z}^{G_z} (-R_{\tilde{C}^z}^{G_z} \varphi_2 \otimes \varphi_3 \otimes \varphi_1), \\ \text{Ind}_{G_z}^G \text{infl}_{G_z}^{G_z} (-R_{\tilde{C}^z}^{G_z} \varphi_3 \otimes \varphi_1 \otimes \varphi_2) \end{array} \right\}$ (φ_i distinct) (Prop. 4.5)	$\text{Ind}_{\tilde{Z}\tilde{G}_y}^{\tilde{G}} \text{infl}_{\tilde{G}_y}^{\tilde{Z}\tilde{G}_y} (-R_{\tilde{C}}^{\tilde{G}_y} \tilde{\varphi}_1 \otimes \tilde{\varphi}_2 \otimes \tilde{\varphi}_3)$ (Prop. 5.7)

TABLE 1. Depth-zero A-packets for $U(2, 1)$, and their base change lifts. To obtain L -packets, omit the representations marked with an asterisk (*).

restriction of scalars. Similarly, if \underline{G} is a k_F -group, $\tilde{\underline{G}}$ will denote $R_{k_E/k_F}(\underline{G})$. Whenever we use this notation, the extension E/F will either be specified, or it will be understood from the context.

For every nonarchimedean local field F and every reductive algebraic F -group \underline{G} , one has an associated *extended affine building* $\mathcal{B}(\underline{G}, F)$, as defined by Bruhat and Tits [3, 4]. As a G -set, $\mathcal{B}(\underline{G}, F)$ is a direct product of an affine space (on which G acts via translation) and the *reduced building* $\mathcal{B}^{\text{red}}(\underline{G}, F)$, which depends only on $\underline{G}/\underline{Z}$, where \underline{Z} is the center of \underline{G} . Note that Z fixes $\mathcal{B}^{\text{red}}(\underline{G}, F)$. For any extension E/F of finite residue degree, $\mathcal{B}(\underline{G}, F)$ always has a natural embedding into $\mathcal{B}(\tilde{\underline{G}}, F) = \mathcal{B}(\underline{G}, E)$. To every point $x \in \mathcal{B}(\underline{G}, F)$, there is an associated parahoric subgroup G_x of G . The stabilizer of x in G contains G_x with finite index. The pro- p -radical of G_x is denoted G_{x+} , and the quotient G_x/G_{x+} is the group of rational points of a connected reductive k_F -group \underline{G}_x . These objects depend only on the image of x in $\mathcal{B}^{\text{red}}(\underline{G}, F)$. Thus, in the case of a torus \underline{T} , we may write T_0 , T_{0+} , and \underline{T} instead of T_x , T_{x+} , and \underline{T}_x , since these do not depend on the choice of x . More generally, G_{0+} will denote the set of topologically unipotent elements in G .

We now present an elementary fact about the building that we will use several times throughout this paper.

Lemma 2.1. *Let \underline{Z} denote the center of \underline{G} , and let $y, z \in \mathcal{B}(\underline{G}, F)$ have distinct images in $\mathcal{B}^{\text{red}}(\underline{G}, F)$. Suppose G_y is a maximal parahoric subgroup, $\gamma \in G_y$, and the image $\bar{\gamma}$ of γ in \underline{G}_y is regular elliptic (i.e., $\bar{\gamma}$ belongs to no proper k_F -parabolic subgroup of \underline{G}_y). Then $\gamma \notin ZG_z$.*

Proof. First, suppose $\gamma \in ZG_z \setminus G_z$. From [10, Lemma 4.2.1], γ does not fix z . Therefore, γ must act on some line containing z via a nontrivial translation. By [10, Cor. 3.1.5], γ cannot fix y , a contradiction.

Now suppose $\gamma \in G_z$. Then $\gamma \in G_x$ for all x lying on the geodesic between y and z . For such an x that is close to but not equal to y , G_x is a subgroup of G_y , and the image of G_x in \underline{G}_y is the group of k_F -fixed points of a proper parabolic subgroup. Thus $\gamma \notin G_x$, a contradiction, and the lemma follows. \square

If \underline{G} is a connected reductive group over a finite field, \underline{T} is a maximal torus in \underline{G} , and θ is a (complex) character of \underline{T} , then let $R_{\underline{T}}^{\underline{G}}\theta$ denote the corresponding Deligne-Lusztig virtual character of \underline{G} [12].

For any reductive algebraic group \underline{G} defined over a local or finite field, we have the following notation.

- $\mathbf{1}_G$ will denote the trivial representation of G .
- St_G will denote the Steinberg representation of G .
- For any character ψ of G , $\text{St}_G(\psi)$ will denote $\text{St}_G \cdot \psi$.
- For any representation σ of a subgroup H of G , $\text{ind}_H^G \sigma$ will denote the representation of G obtained from σ via normalized compact induction.
- If \underline{Z} is the center of \underline{G} and ω is a character of \underline{Z} , then let $C(G, \omega)$ denote the space of complex-valued, locally constant functions f on G such that the support of f is compact modulo Z , and $f(gz) = f(g)\omega(z)$ for all $g \in G$ and $z \in Z$.
- G^{reg} denotes the set of regular semisimple elements of G .
- For any admissible, finite-length representation π of G , let θ_π denote the character of π , considered either as a function on the set of elements or

conjugacy classes of G (of G^{reg} in the local-field case), or as a distribution on an appropriate function space on G .

- Suppose ε is an automorphism of G . Then ε acts in a natural way on the set of equivalence classes of irreducible, admissible representations of G . Suppose π is such a representation and $\pi \cong \pi^\varepsilon$. Let $\pi(\varepsilon)$ denote an intertwining operator from π to π^ε . If ε has order ℓ , then we can and will normalize $\pi(\varepsilon)$ by requiring that the scalar $\pi(\varepsilon)^\ell$ equal 1. Then $\pi(\varepsilon)$ is well determined up to a scalar ℓ th root of unity. The ε -twisted character of π is the distribution $\theta_{\pi, \varepsilon}$ defined by $\theta_{\pi, \varepsilon}(f) = \text{trace}(\pi(f)\pi(\varepsilon))$ for $f \in C_c^\infty(G)$. As with the character, the twisted character can be represented by a function (again denoted $\theta_{\pi, \varepsilon}$) on G (G^{reg} in the local-field case). We may regard $\theta_{\pi, \varepsilon}$ as a function on the set of ε -twisted conjugacy classes.

Note that $\theta_{\pi, \varepsilon}$ still makes sense when π is an admissible, finite-length representation.

- For any maximal torus \underline{T} of \underline{G} , let $W(T, G)$ denote the quotient of T in its normalizer in G , and let $W_F(\underline{T}, \underline{G})$ denote the group of F -points of the absolute Weyl group $N_{\underline{G}}(\underline{T})/\underline{T}$.

2.2. Shintani lifting. Suppose that E/F is a finite, cyclic extension of local or finite fields, $\Gamma = \text{Gal}(E/F)$, and \underline{G} is a connected reductive algebraic F -group. Let ε denote a generator of Γ , and let ℓ denote the order of Γ . Then one can define a norm mapping from $\tilde{\underline{G}}$ to \underline{G} by

$$x \mapsto x \cdot \varepsilon(x) \cdot \dots \cdot \varepsilon^{\ell-1}(x).$$

If x is defined over F then, in general, the most that one can say about the image of x is that its conjugacy class in \underline{G} is defined over F . If F is local and \underline{G} has a simply connected derived group, then such a conjugacy class must have F -points [23]. Thus, an F -point $x \in \tilde{\underline{G}}$ determines a stable conjugacy class in G . Any stable, ε -twisted conjugate of x determines the same stable conjugacy class in G . Thus, we have a map $\mathcal{N}_{E/F}^{\tilde{\underline{G}}}$ from the set of stable, ε -twisted conjugacy classes of $\tilde{\underline{G}}$ to the set of stable conjugacy classes in G . If x commutes with its Galois conjugates, then we may and will define $\mathcal{N}_{E/F}^{\tilde{\underline{G}}}(x) \in G$ via the formula above.

Call $g \in \tilde{\underline{G}}$ ε -regular if $\mathcal{N}(g)$ is regular. Let $\tilde{G}^{\varepsilon\text{-reg}}$ denote the set of ε -regular elements.

If Π and $\tilde{\Pi}$ are finite sets of representations of G and \tilde{G} , respectively, we say that $\tilde{\Pi}$ is the *Shintani lift* (or *base change*) of Π if

$$\Theta_{\tilde{\Pi}, \varepsilon}(g) = \Theta_{\Pi}(\mathcal{N}(g))$$

for all $g \in \tilde{G}$ (all $g \in \tilde{G}^{\varepsilon\text{-reg}}$ in the local-field case), where Θ_{Π} and $\Theta_{\tilde{\Pi}, \varepsilon}$ are non-trivial stable (resp. ε -stable) linear combinations of the characters (resp. ε -twisted characters) of the elements of Π (resp. $\tilde{\Pi}$).

If \underline{T} is an F -torus then for any character λ of T , define the character $\tilde{\lambda}$ of \tilde{T} by $\tilde{\lambda} = \lambda \circ \mathcal{N}_{E/F}^{\underline{T}}$.

2.3. Notation related to unitary groups. From now on, fix a nonarchimedean local field F of characteristic zero with finite residue field k_F of odd order q . Let E be the unramified quadratic extension of F . Let E^1 (resp. k_E^1) denote the kernel

of the norm from E to F (resp. k_E to k_F). Let ε denote the nontrivial element of the Galois group $\Gamma = \text{Gal}(E/F)$.

Let \underline{G} denote a unitary group in three variables defined with respect to E/F . Then \underline{G} is uniquely determined up to isomorphism, and we can and will assume that \underline{G} is the unitary group defined by the Hermitian matrix

$$\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then

$$G = \underline{G}(F) = \{g \in \text{GL}(3, E) : g \Phi {}^t(\varepsilon(g)) = \Phi\}$$

and $\tilde{G} = \text{GL}(3, E)$. Let \underline{G} denote the corresponding algebraic group over k_F . Then $\tilde{G} = \text{GL}(3, k_E)$.

Let \underline{Z} denote the center of \underline{G} . So, following our notational conventions, \tilde{Z} is the center of \tilde{G} .

Let $\mathcal{B} = \mathcal{B}(\underline{G}, F)$ and $\tilde{\mathcal{B}} = \mathcal{B}(\tilde{G}, F) = \mathcal{B}(\underline{G}, E)$. Note that ε acts on $\tilde{\mathcal{B}}$, and we may and will identify the set of fixed points $\tilde{\mathcal{B}}^\varepsilon$ with \mathcal{B} .

Since \underline{G} is F -quasisplit, it contains F -Borel subgroups. In particular, $\tilde{\mathcal{B}}$ must contain some ε -invariant apartment $\tilde{\mathcal{A}}$ with more than one ε -fixed point. Choose an ε -fixed point y in an ε -invariant minimal facet in $\tilde{\mathcal{A}}$, and an ε -invariant alcove $\tilde{\mathcal{F}}$ in $\tilde{\mathcal{A}}$, such that the closure of $\tilde{\mathcal{F}}$ contains y . (Let \mathcal{F} denote the set of ε -fixed points of $\tilde{\mathcal{F}}$.) Then these choices determine an F -Borel subgroup \underline{B} together with a Levi factor \underline{M} of \underline{B} . Note that M is isomorphic to $E^\times \times E^1$. We may assume that our choices of y and $\tilde{\mathcal{F}}$ allow us to realize \underline{B} explicitly as the group of upper triangular matrices in \underline{G} , and \underline{M} as the group of diagonal matrices.

The boundary of \mathcal{F} contains two points: the previously chosen point y , and another point that we will denote z . Note that $\tilde{\mathcal{F}}$ is the direct product of a one-dimensional affine space and an ε -invariant equilateral triangle Δ in the reduced building of \tilde{G} (which we will identify with a subset of $\tilde{\mathcal{B}}$), y the ε -fixed vertex of Δ , and z is the midpoint of the wall of Δ that is opposite y . In \mathcal{B} , y and z are both vertices, but only y is hyperspecial.

Consider the map $\lambda: U(1) \rightarrow \underline{G}$ given by $t \mapsto \text{diag}(1, t, 1)$. Since $\widetilde{U(1)} \cong \text{GL}(1)$, we actually have a one-parameter subgroup of \tilde{G} . In the usual way, λ determines a parabolic F -subgroup $\tilde{P} = \tilde{P}_\lambda$ of \tilde{G} , together with a Levi decomposition of \tilde{P} . Let \tilde{H} denote the corresponding Levi factor. Then \tilde{H} is the group of invertible matrices of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}.$$

This subgroup arises via restriction of scalars from a subgroup \underline{H} of \underline{G} . Note that \underline{H} is an E -Levi, but not F -Levi, subgroup of \underline{G} . It is an endoscopic group for \underline{G} , isomorphic to $U(1, 1) \times U(1)$.

Similarly, we can define a subgroup \underline{H} of \underline{G} and a parabolic k_F -subgroup \tilde{P} of \tilde{G} with Levi factor \tilde{H} . Note that $\underline{G}_y \cong \underline{G}$ and $\underline{G}_z \cong \underline{H}$.

Up to conjugacy, \underline{H} contains two F -tori that are isomorphic to $U(1) \times U(1) \times U(1)$. The group of F -points of one of these tori fixes a hyperspecial vertex, and the group of F -points of the other fixes a non-hyperspecial vertex. Pick such a torus whose F -points fix y (resp. z) and call it \underline{C} (resp. \underline{C}'). Given the right choices, we can

and will realize \underline{C} as the set of matrices of the form

$$\gamma = \begin{pmatrix} \frac{\gamma_1 + \gamma_3}{2} & 0 & \frac{\gamma_1 - \gamma_3}{2} \\ 0 & \gamma_2 & 0 \\ \frac{\gamma_1 - \gamma_3}{2} & 0 & \frac{\gamma_1 + \gamma_3}{2} \end{pmatrix}$$

where $\gamma_i \in U(1)$. We define the torus $\underline{C} \subset \underline{G}$ similarly. We identify \underline{C} (and similarly \underline{C}') with $U(1) \times U(1) \times U(1)$ via the map $\gamma \mapsto (\gamma_1, \gamma_2, \gamma_3)$. We will realize \underline{C}' as $\nu \underline{C} \nu^{-1}$, where

$$\nu = \begin{pmatrix} 1/\sqrt{\varpi_F} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\varpi_F} \end{pmatrix}.$$

(The ambiguity in the choice of square root of ϖ_F has no effect.)

Let \underline{B}_y (resp. \underline{B}_z) denote the Borel subgroup of \underline{G}_y (resp. \underline{G}_z) determined by \mathcal{F} .

For any F -group \underline{L} , let $\mathcal{N}^{\underline{L}} = \mathcal{N}_{E/F}^{\underline{L}}$. When $\underline{L} = \underline{G}$, we simply write \mathcal{N} . Similarly, for any k_F -group \underline{L} , let $\mathcal{N}^{\underline{L}} = \mathcal{N}_{k_E/k_F}^{\underline{L}}$. When $\underline{L} = \underline{G}$, we simply write $\tilde{\mathcal{N}}$.

For any subgroup $\underline{S} \subset \tilde{\underline{G}}$, let $\det_{\underline{S}}$ denote the restriction of the determinant to \underline{S} . We will omit the subscript when it is clear from the context. Similar notation holds for subgroups of $\tilde{\underline{G}}$.

2.4. Cartan subgroups of \underline{G} . For a quadratic extension L/K , denote by $U(1, L/K)$ the unitary group in one variable over K defined with respect to L/K . Up to stable conjugacy, there are four kinds of Cartan subgroup of \underline{G} . In the notation of [23], they are isomorphic to:

$$(2.4-0) \quad R_{E/F}(\mathrm{GL}(1)) \times U(1, E/F),$$

$$(2.4-1) \quad U(1, E/F) \times U(1, E/F) \times U(1, E/F),$$

$$(2.4-2) \quad R_{E/F}(U(1, EK/K)) \times U(1, E/F) \text{ for } K \text{ a ramified quadratic extension of } F,$$

$$(2.4-3) \quad R_{L/F}(U(1, EL/L)) \text{ for } L \text{ a cubic extension of } F.$$

2.5. Representation theory of \mathbf{G} , \mathbf{H} , and \mathbf{C} . A reference for much of this section is [27].

Representations of \mathbf{G} . Let \mathbf{B} denote a Borel subgroup of \mathbf{G} with Levi factor \mathbf{M} , and let θ be a character of \mathbf{M} . Then the induced representation $\mathrm{ind}_{\mathbf{B}}^{\mathbf{G}} \theta$ is irreducible except when θ extends to a character of \mathbf{H} . In this case, the induced representation is a sum of two irreducible components. If θ extends to a character θ_0 of \mathbf{G} , then these components are θ_0 and $\mathrm{St}_{\mathbf{G}}(\theta_0)$.

Let L denote a cubic unramified extension of E . Then \mathbf{G} contains a torus \mathbf{S} that is isomorphic to the kernel of the norm map from k_{EL} to k_L . Let \mathbf{T} be either \mathbf{S} or \mathbf{C} . For any character θ of \mathbf{T} with trivial stabilizer in $W_{k_F}(\underline{\mathbf{T}}, \underline{\mathbf{G}})$, we have a Deligne-Lusztig cuspidal representation whose character is $-R_{\mathbf{T}}^{\mathbf{G}} \theta$. For $\mathbf{T} = \mathbf{S}$, we will call such representations “cubic cuspidal representations.”

The other irreducible representations of \mathbf{G} have the form $\tau \cdot \psi$, where τ is the cuspidal unipotent representation and ψ is a character.

Representations of H . As above let B be a Borel subgroup of G with Levi factor M and let θ be a character of M . The induced representation $\text{ind}_{B \cap H}^H \theta$ is irreducible except when θ extends to a character θ_1 of H . In this case, the induced representation is the sum of θ_1 and $\text{St}_H(\theta_1)$.

The remaining representations of H are the Deligne-Lusztig cuspidal representations, whose characters are of the form $-R_{\mathbb{C}}^G \theta$ for $\theta \in \text{Hom}(\mathbb{C}, \mathbb{C}^\times)$ in general position with respect to the action of $W_{k_F}(\underline{\mathbb{C}}, \underline{H})$.

Representations of C . We will need a technical result on linear combinations of characters of C . Let A be a finite abelian group of order n , and let χ_1, \dots, χ_n be the irreducible characters of A . The following three lemmas concern characters of products of copies of A .

Lemma 2.2. *Let*

$$f = \sum_{i=1}^n a_i \chi_i,$$

where $a_i \in \mathbb{C}$. Suppose that f vanishes off of a subset of A of size 2. Then either $f = 0$, or the number of i such that $a_i \neq 0$ is at least $n/2$.

Proof. Let $\{a, b\}$ be the above subset of A . We have

$$na_i = n \cdot \langle f, \chi_i \rangle = f(a) \bar{\chi}_i(a) + f(b) \bar{\chi}_i(b).$$

Assume $f \neq 0$. If $f(b) = 0$, then for all i , $a_i = f(a) \bar{\chi}_i(a)/n \neq 0$. If $f(b) \neq 0$, then $a_i \neq 0$ unless $\bar{\chi}_i(ba^{-1}) = -f(a)/f(b)$. Since $ba^{-1} \neq 1$, this equality holds for at most $n/2$ values of i . \square

Lemma 2.3. *Let N be the subset of $A \times A$ consisting of all elements (a, b) such that $a \neq b$. Suppose that for some $a_{ij} \in \mathbb{C}$,*

$$f = \sum_{i,j} a_{ij} \chi_i \otimes \chi_j$$

vanishes on N . Then either $f = 0$ or at least n of the a_{ij} are nonzero.

Proof. Assume $f \neq 0$. Fix $a \in A$. Evaluating f at (a, b) for $b \neq a$, we obtain that the function

$$\sum_j \left(\sum_i a_{ij} \chi_i(a) \right) \chi_j$$

on A vanishes on $A - \{a\}$. It follows easily that either this function vanishes on A , or for all j , the coefficient $\sum_i a_{ij} \chi_i(a)$ is nonzero. The former case cannot happen since $f \neq 0$. In the latter case, it follows that for all j , at least one coefficient a_{ij} must be nonzero. Hence at least n of the a_{ij} must be nonzero. \square

Lemma 2.4. *Let N' be the subset of $A \times A \times A$ consisting of all elements (a, b, c) such that a, b , and c are distinct. Suppose that for some $a_{ijk} \in \mathbb{C}$,*

$$f = \sum_{i,j,k} a_{ijk} \chi_i \otimes \chi_j \otimes \chi_k$$

vanishes on N' . Then either f vanishes on $A \times A \times A$ or at least $n/2$ of the a_{ijk} are nonzero.

Proof. Assume $f \neq 0$. Fix $a \neq b$ in A . Then the function

$$\sum_k \left(\sum_{i,j} a_{ijk} \chi_i(a) \chi_j(b) \right) \chi_k$$

on A vanishes off of $\{a, b\}$. Hence, by Lemma 2.2, either this function vanishes on A , or for at least $n/2$ values of k , the coefficient $\sum_{i,j} a_{ijk} \chi_i(a) \chi_j(b)$ is nonzero. In the latter case, for each such k , at least one coefficient a_{ijk} must be nonzero. Hence at least $n/2$ of the a_{ijk} are nonzero.

We may therefore assume that the former case holds for all pairs $a \neq b$. By the linear independence of characters, the coefficient $\sum_{i,j} a_{ijk} \chi_i(a) \chi_j(b)$ must vanish for all k and all pairs $a \neq b$. Since $f \neq 0$, $a_{i'j'k'} \neq 0$ for some i', j', k' . Thus the function $\sum_{i,j} a_{ijk'} \chi_i \otimes \chi_j$ on $A \times A$ vanishes on the set N of Lemma 2.3, but it does not vanish on $A \times A$ since $a_{i'j'k'} \neq 0$. Hence Lemma 2.3 implies that at least n of the coefficients $a_{ijk'}$ must be nonzero. \square

Corollary 2.5. *Suppose that*

$$\sum_{\chi \in \text{Hom}(C, \mathbb{C}^\times)} a_\chi \chi$$

vanishes on $C \cap G^{\text{reg}}$, where $a_\chi \in \mathbb{C}$. Then either this linear combination vanishes on C or at least $(q+1)/2$ of the a_χ are nonzero. \square

Corollary 2.6. *Suppose that $q > 59$, and let $f = \sum a_\chi \chi$ be a linear combination of at most 30 characters of C . If f vanishes on $C \cap G^{\text{reg}}$, then f vanishes on C .* \square

2.6. Shintani lifting for \mathbf{G} and \mathbf{H} . According to [27], the irreducible characters of \mathbf{G} are of the form $\pm R_{\mathbf{L}}^{\mathbf{G}} \theta$, where \mathbf{L} is the connected centralizer of some semisimple element of \mathbf{G} , and θ is the twist of a unipotent character of \mathbf{L} by a one-dimensional character in general position. Moreover, one obtains a cuspidal character of \mathbf{G} precisely when \mathbf{L} is an elliptic torus or when $\mathbf{L} = \mathbf{G}$ and θ is a twist of the unique cuspidal unipotent character of \mathbf{G} .

By [17], our assumption that k_F has odd characteristic guarantees the existence of Shintani descent from $\tilde{\mathbf{G}}$ to \mathbf{G} . In [14], Digne gives a general proof that Shintani descent is compatible with Deligne-Lusztig induction. In particular, if σ is an irreducible representation of \mathbf{G} with character $\pm R_{\mathbf{L}}^{\mathbf{G}} \theta$ (θ a character of \mathbf{L}), then the character of the Shintani lift $\tilde{\sigma}$ of σ from \mathbf{G} to $\tilde{\mathbf{G}}$ is of the form $\pm R_{\mathbf{L}}^{\tilde{\mathbf{G}}} \tilde{\theta}$, where $\tilde{\theta}$ is the Shintani lift of θ . Now $\tilde{\mathbf{L}}$ is a Levi factor of a parabolic subgroup of $\tilde{\mathbf{G}}$ unless \mathbf{L} is isomorphic to the torus \mathbf{S} defined in §2.5. Hence $\tilde{\sigma}$ is a parabolically induced representation unless $\mathbf{L} \cong \mathbf{S}$ or $\mathbf{L} = \mathbf{G}$. In the former case, $\tilde{\mathbf{S}}$ is an elliptic torus isomorphic to k_{EL}^\times and $\tilde{\sigma}$ is cuspidal. In the latter case, σ is a one-dimensional representation $\varphi \circ \det_{\mathbf{G}}$, a twist $\text{St}_{\mathbf{G}}(\varphi \circ \det_{\mathbf{G}})$ of the Steinberg representation, or a twist $\tau(\varphi \circ \det_{\mathbf{G}})$ of the cuspidal unipotent representation. One shows easily that the Shintani lift $\tilde{\sigma}$ is, respectively, $\tilde{\varphi} \circ \det_{\tilde{\mathbf{G}}}$, $\text{St}_{\tilde{\mathbf{G}}}(\tilde{\varphi} \circ \det_{\tilde{\mathbf{G}}})$, or $\tilde{\tau}(\tilde{\varphi} \circ \det_{\tilde{\mathbf{G}}})$, where $\tilde{\tau}$ is the unipotent representation of $\tilde{\mathbf{G}}$ not equivalent to $\mathbf{1}_{\tilde{\mathbf{G}}}$ or $\text{St}_{\tilde{\mathbf{G}}}$. The remaining representations of \mathbf{G} are those whose characters are of the form $R_{\mathbf{H}}^{\mathbf{G}} \theta$. By [14], the Shintani lifts of such representations are representations induced from $\tilde{\mathbf{P}}$. Hence the cubic cuspidal representations of \mathbf{G} are exactly those irreducible representations of \mathbf{G} whose Shintani lifts are cuspidal.

We now consider Shintani lifting for irreducible representations of \mathbf{H} . From §2.5, most such representations have characters of the form $\pm R_{\mathbf{T}}^{\mathbf{H}}\theta$. From Digne [14], the Shintani lift of such a representation has character $\pm R_{\mathbf{T}}^{\mathbf{H}}\tilde{\theta}$.

The remaining representations of \mathbf{H} are the one-dimensional representations $\varphi \circ \det_{\mathbf{H}}$ and the Steinberg representations $\mathrm{St}_{\mathbf{H}}(\varphi \circ \det_{\mathbf{H}})$. It is easy to see that the respective Shintani lifts of these representations are $\tilde{\varphi} \circ \det_{\mathbf{H}}$ and $\mathrm{St}_{\mathbf{H}}(\tilde{\varphi} \circ \det_{\mathbf{H}})$.

2.7. Depth-zero representations of G .

Principal series of G . For $\lambda \in \mathrm{Hom}(M, \mathbb{C}^\times)$, there exist unique characters $\lambda_1 \in \mathrm{Hom}(E^\times, \mathbb{C}^\times)$ and $\lambda_2 \in \mathrm{Hom}(E^1, \mathbb{C}^\times)$ such that

$$(2.7.1) \quad \lambda \left(\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \bar{\alpha}^{-1} \end{pmatrix} \right) = \lambda_1(\alpha)\lambda_2(\alpha\bar{\alpha}^{-1}\beta),$$

where $\alpha \in E^\times$, $\beta \in E^1$. By [18], $\mathrm{ind}_B^G \lambda$ is irreducible except for in the following cases:

- (2.7PS-1) $\lambda_1 = |\cdot|_E^{\pm 1}$
- (2.7PS-2) $\lambda_1|_{F^\times} = \omega_{E/F}|\cdot|_F^{\pm 1}$
- (2.7PS-3) λ_1 is nontrivial and $\lambda_1|_{F^\times}$ is trivial.

In case (2.7PS-1), $\mathrm{ind}_B^G \lambda$ has two constituents: the one-dimensional representation $\psi = \lambda_2 \circ \det$, and the square-integrable Steinberg representation $\mathrm{St}_G(\psi)$.

In case (2.7PS-2), $\mathrm{ind}_B^G \lambda$ also has two constituents: a square-integrable representation $\pi^2(\lambda)$ and a non-tempered unitary representation $\pi^n(\lambda)$.

In case (2.7PS-3), $\mathrm{ind}_B^G \lambda$ decomposes into a direct sum $\pi_1(\lambda) \oplus \pi_2(\lambda)$.

By [21], $\mathrm{ind}_B^G \lambda$ has depth zero if and only if λ has depth zero.

Other representations of G . Since G has no non-minimal proper parabolic subgroups, the remaining irreducible representations are all supercuspidal. From either [21] or [20], we know that all such representations have a unique expression of the form $\mathrm{ind}_{G_x}^G \sigma$, where $x = y$ or z , and σ is the inflation to G_x of an irreducible cuspidal representation $\bar{\sigma}$ of \mathbf{G}_x . The representations $\bar{\sigma}$ are classified in §2.5. Based on this classification, we have the following kinds of supercuspidal representation of depth zero.

- (2.7SC-1) $\mathrm{ind}_{G_y}^G \sigma$, where $\bar{\sigma}$ is a cubic cuspidal representation of $\mathbf{G}_y \cong \mathbf{G}$.
- (2.7SC-2) $\mathrm{ind}_{G_y}^G \sigma$, where $\bar{\sigma}$ is a cuspidal representation of \mathbf{G}_y with character $-R_{\mathbf{C}}^{\mathbf{G}_y} \varphi$ and $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ is a regular character of \mathbf{C} (with respect to $W_{k_F}(\underline{\mathbf{C}}, \underline{\mathbf{G}}_y)$).
- (2.7SC-3) $\mathrm{ind}_{G_y}^G \sigma$, where $\bar{\sigma}$ is the twist $\tau \cdot (\eta \circ \det)$ of the cuspidal unipotent representation τ of \mathbf{G}_y , and $\eta \in \mathrm{Hom}(k_E^1, \mathbb{C}^\times)$.
- (2.7SC-4) $\mathrm{ind}_{G_z}^G \sigma$, where $\bar{\sigma}$ is a cuspidal representation of $\mathbf{G}_z \cong \mathbf{H}$ with character $-R_{\mathbf{C}'}^{\mathbf{G}_z} \varphi$ and $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ is a regular character of \mathbf{C}' (with respect to $W_{k_F}(\underline{\mathbf{C}'}, \underline{\mathbf{G}}_z)$). Recall that, according to our notational conventions, $\underline{\mathbf{C}'}$ is the finite reductive quotient of the (unique) parahoric subgroup of $\underline{\mathbf{C}'}$.

3. DESCRIPTION OF DEPTH-ZERO L -PACKETS AND EXPLICIT BASE CHANGE FOR UNITARY GROUPS IN TWO VARIABLES

In this section we give brief descriptions of the depth-zero L -packets for the quasi-split group $U(1, 1)(F)$ and the compact group $U(2)(F)$, as well as their base change lifts to $\mathrm{GL}_2(E)$. We omit the proofs as they are entirely analogous to (but less complicated than) those for $U(2, 1)$. Let \underline{H}^0 be the group $U(1, 1)$, which we will view as the subgroup of \underline{H} consisting of all matrices of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix}.$$

For every subgroup \underline{L} of \underline{G} , let \underline{L}^0 denote the subgroup $\underline{L} \cap \underline{H}^0$ of \underline{H}^0 . Let \underline{H}^1 denote the compact inner form $U(2)$ of H^0 . Note that $\tilde{\underline{H}}^0(F) \cong \tilde{\underline{H}}^1(F) \cong \mathrm{GL}_2(E)$.

From our descriptions, it will be clear that the analogue of Theorem 1.1 holds for unitary groups in two variables.

3.1. Depth-zero L -packets for $U(1, 1)$. The L -packets of H^0 are the $\mathrm{PGL}_2(F)$ -orbits on the set of equivalence classes of irreducible admissible representations of H^0 [23, §11.1]. We first describe the principal series L -packets.

Let $\lambda \in \mathrm{Hom}(M^0, \mathbb{C}^\times) = \mathrm{Hom}(E^\times, \mathbb{C}^\times)$. According to [23, §11.1], the principal series $\mathrm{ind}_{B^0}^{H^0} \lambda$ is irreducible except in the cases

- (1) $\lambda|_{F^\times} = |\cdot|_F^{\pm 1}$
- (2) $\lambda|_{F^\times} = \omega_{E/F}$.

In the first case, $\mathrm{ind}_{B^0}^{H^0} \lambda$ has two constituents: the one-dimensional representation $\psi = \mu \circ \det$, where $\mu \circ \mathcal{N} = \lambda|_E \cdot |\cdot|_E^{-1/2}$, and the Steinberg representation $\mathrm{St}_G(\psi)$. In the second case, $\mathrm{ind}_{B^0}^{H^0} \lambda$ decomposes into a direct sum $\pi_1(\lambda) \oplus \pi_2(\lambda)$ of irreducible representations. By [21], $\mathrm{ind}_{B^0}^{H^0} \lambda$ has depth zero if and only if λ has depth zero.

The principal series L -packets of G are as follows [23, §11.1]. (Here λ and ψ denote one-dimensional representations of M^0 and H^0 , respectively.)

- (1) $\{\mathrm{ind}_{B^0}^{H^0} \lambda\}$, where $\mathrm{ind}_{B^0}^{H^0} \lambda$ is irreducible;
- (2) $\{\psi\}$;
- (3) $\{\mathrm{St}_{H^0}(\psi)\}$;
- (4) $\{\pi_1(\lambda), \pi_2(\lambda)\}$, where $\mathrm{ind}_{B^0}^{H^0} \lambda$ is reducible of the second type described above.

The remaining irreducible representations and L -packets of G are all supercuspidal. The depth zero supercuspidals of H^0 have a unique expression of the form $\mathrm{ind}_{H_v^0}^{H^0} \sigma$, where $v = y$ or z , and σ is the inflation to H_v^0 of an irreducible cuspidal representation $\bar{\sigma}$ of H_v^0 . Let \underline{C}_v^0 be $\underline{C} \cap \underline{H}^0$ if $v = y$, and $\underline{C}' \cap \underline{H}^0$ if $v = z$. Then the character of such a representation $\bar{\sigma}$ must be of the form $-R_{\underline{C}_v^0}^{H_v^0} \varphi$, where φ is a character of \underline{C}_v^0 in general position. Since $\underline{C}_v^0 \cong k_E^1 \times k_E^1$, we may view any such character as having the form $\varphi_1 \otimes \varphi_2$, where the φ_i are distinct characters of k_E^1 .

Fix a cuspidal representation $\bar{\sigma}$ of H^0 . Viewing it as a representation of H_v^0 , we inflate it to a representation σ_v of H_v^0 . Let $\pi_v = \mathrm{ind}_{H_v^0}^{H^0} \sigma_v$. Then $\{\pi_y, \pi_z\}$ is a depth-zero supercuspidal L -packet of H^0 . Conversely, all such L -packets are of this form. If π_y and π_z are formed from the character $\varphi_1 \otimes \varphi_2$ of \underline{C}^0 as above, then for future reference call this L -packet $\Pi_{\varphi_1, \varphi_2}^0$.

3.2. Base change lifts for $U(1, 1)$. By [23, §11.4], the base change lifts of principal series L -packets of H^0 are as follows. Let $\lambda \in \text{Hom}(M^0, \mathbb{C}^\times)$.

- (i) If $\text{ind}_{B^0}^{H^0} \lambda$ is irreducible and $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$ is irreducible, then the base change lift of the L -packet $\{\text{ind}_{B^0}^{H^0} \lambda\}$ is $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$.
- (ii) If $\text{ind}_{B^0}^{H^0} \lambda$ is irreducible but $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$ is reducible, then $\lambda|_{F^\times} = |\cdot|_F^{\pm 1} \omega_{E/F}$, and the base change lift of the L -packet $\{\text{ind}_{B^0}^{H^0} \lambda\}$ is $\lambda| \cdot |_E^{\mp 1/2} \circ \det$.
- (iii) If $\lambda|_{F^\times} = |\cdot|_F^{\pm 1}$, let ψ be the one-dimensional representation $\mu \circ \det_{\underline{H}^0}$, where $\mu \circ \mathcal{N} = \lambda| \cdot |_E^{\mp 1/2}$. Then the lift of the L -packet consisting of the constituent ψ (resp., the Steinberg constituent $\text{St}_{H^0}(\psi)$) of $\text{ind}_{B^0}^{H^0} \lambda$ is the one-dimensional constituent $\tilde{\psi} = (\lambda| \cdot |_E^{\mp 1/2}) \circ \det_{\tilde{H}^0}$ (resp., the Steinberg constituent $\text{St}_{\tilde{H}^0}(\tilde{\psi})$) of $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$.
- (iv) If $\lambda|_{F^\times} = \omega_{E/F}$, then the lift of the L -packet $\{\pi_1(\lambda), \pi_2(\lambda)\}$ is $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \tilde{\lambda}$.

The base change lift of the depth-zero supercuspidal L -packet $\Pi_{\varphi_1, \varphi_2}^0$ is the principal series representation $\text{ind}_{\tilde{B}^0}^{\tilde{H}^0} \varphi^*$, where φ^* is the character $\hat{\varphi}_1 \omega_{E'/E} \otimes \hat{\varphi}_2 \omega_{E'/E}$ of $E^\times \times E^\times \cong \tilde{M}^0$. Here, E' is an unramified quadratic extension of E , and $\hat{\varphi}_i$ is the inflation to E^\times of the character $\tilde{\varphi}_i$ of k_E^\times .

3.3. Depth-zero L -packets for $U(2)$. Since H^1 is compact, it has only one parabolic subgroup (and in fact is equal to it). The finite reductive quotient H^1 is isomorphic to $k_E^1 \times k_E^1$. Thus, every irreducible, depth-zero representation of H^1 has the form $\text{infl}(\varphi_1 \otimes \varphi_2)$, the inflation to H^1 of a character of H^1 .

Let

$$\Pi_{\varphi_1, \varphi_2}^1 = \begin{cases} \{\text{infl}(\varphi_1 \otimes \varphi_2), \text{infl}(\varphi_2 \otimes \varphi_1)\} & \text{if } \varphi_1 \neq \varphi_2, \\ \{\text{infl}(\varphi_1 \otimes \varphi_2)\} & \text{if } \varphi_1 = \varphi_2. \end{cases}$$

Then we declare the $\Pi_{\varphi_1, \varphi_2}^1$ to be the L -packets for H^1 . These L -packets are chosen so as to make the correspondence JL given in §3.4 work properly.

3.4. Base change lifts for $U(2)$ via a Jacquet-Langlands-like correspondence. Since \underline{H}^1 is an inner form of \underline{H}^0 , we can obtain a base change lift if we can associate each L -packet for H^1 to one for H^0 . This association will be similar to the Jacquet-Langlands correspondence (or “Abstract Matching Theorem” [11, 22, 2]). That is, given an L -packet Π^1 for H^1 , we want to find an L -packet Π^0 for H^0 such that

$$(3.4.1) \quad \sum_{\pi \in \Pi^1} \theta_\pi(g_1) = \pm \sum_{\pi \in \Pi^0} \theta_\pi(g_0)$$

for all regular $g_1 \in H^1$ and $g_0 \in H^0$ whose stable conjugacy classes are associated in a natural way.

Define a map JL from the depth-zero L -packets of H^1 to those of H^0 by

$$JL(\Pi_{\varphi_1, \varphi_2}^1) = \Pi_{\varphi_1, \varphi_2}^0$$

if $\varphi_1 \neq \varphi_2$. If $\varphi = \varphi_1 = \varphi_2$, then we define $JL(\Pi_{\varphi_1, \varphi_2}^1)$ as follows. Form the character $\varphi \circ \mathcal{N}$ of k_E^\times , which we can then inflate to a character λ of E^\times . Now let $JL(\Pi_{\varphi_1, \varphi_2}^1)$ be the Steinberg component of $\text{ind}_{B^0}^{H^0} \lambda| \cdot |_F$. More specifically, this representation is $\text{St}_{H^0}(\mu \circ \det)$, where $\mu \circ \mathcal{N} = \lambda| \cdot |_F$, as in §3.1.

It is not difficult to see that JL is the only correspondence that satisfies (3.4.1) for all $g_1 \in H^1$ whose image in H^1 is regular. Thus, if we assume that there is a Jacquet-Langlands-like correspondence from the depth-zero L -packets of H^0 to those of H^1 , then it must be JL .

4. DESCRIPTION OF DEPTH-ZERO L -PACKETS AND A -PACKETS FOR G

In almost all cases, L -packets and A -packets are the same. In one case (see below), a certain principal series L -packet is enlarged to form an A -packet. Thus, while the L -packets constitute a partition of the set of equivalence classes of irreducible representations, the A -packets do not.

4.1. L -packets consisting of principal series constituents. The following proposition is due to Rogawski [23, §12.2].

Proposition 4.1. *The L -packets of G that consist entirely of principal series constituents all have one of the following forms (where λ and ψ denote one-dimensional representations of M and G , respectively):*

- (4.1-1) $\{\text{ind}_G^P \lambda\}$, where $\text{ind}_G^P \lambda$ is irreducible;
- (4.1-2) $\{\psi\}$;
- (4.1-3) $\{\text{St}_G(\psi)\}$;
- (4.1-4) $\{\pi_1(\lambda), \pi_2(\lambda)\}$, where $\text{ind}_G^P \lambda$ is reducible of type (2.7PS-3).
- (4.1-5) $\{\pi^n(\lambda)\}$, where $\text{ind}_G^P \lambda$ is reducible of type (2.7PS-2).

In the last case, $\pi^n(\lambda)$ is contained in the A -packet $\Pi(\lambda) = \{\pi^n(\lambda), \pi^s(\lambda)\}$, where $\pi^s(\lambda)$ is the supercuspidal representation that sits inside an L -packet with the square-integrable principal series constituent $\pi^2(\lambda)$. In the depth-zero setting, the representation $\pi^s(\lambda)$ will be explicitly described in §4.3

4.2. Singleton supercuspidal L -packets. In this section, we characterize the stable supercuspidal representations of G of depth zero in terms of inducing data.

Proposition 4.2. *A supercuspidal representation π of G of depth zero is stable if and only if π is of the form $\text{ind}_{G_y}^G \sigma$, where σ is the inflation to G_y of a cubic cuspidal representation $\bar{\sigma}$ of G_y .*

Proof. Let π be a representation of the above form. Let γ be an element of G^{reg} and let γ' be a stable conjugate of γ . We will show that

$$(4.2.1) \quad \theta_\pi(\gamma) = \theta_\pi(\gamma').$$

The conjugacy classes contained within the stable conjugacy class of γ are parametrized by

$$\text{Ker}\{H^1(F, \underline{G}_\gamma) \rightarrow H^1(F, \underline{G})\}$$

(see [23, §3.1]), where \underline{G}_γ is the centralizer of γ in \underline{G} . If γ is contained in a Cartan subgroup of G of type (2.4-0) or (2.4-3), then this kernel is trivial by [23, §3.6] so any stable conjugate γ' of γ is a conjugate of γ . Hence $\theta_\pi(\gamma) = \theta_\pi(\gamma')$. Therefore, we may assume that γ is contained in a Cartan subgroup T of type (2.4-1) or (2.4-2).

For any regular, depth-zero X in the dual of the Lie algebra of a cubic torus in G , the germ $\theta_\pi|_{G_{0+}}$ coincides with a constant multiple of the Fourier transform of the orbital integral corresponding to X . (This follows from Corollaire III.10 and Proposition III.8 of [29]. It also follows from the proof of the main theorem of [1].)

The Weyl group of a cubic torus acts via the Galois group, so two regular elements of the torus are conjugate if and only if they are stably conjugate. Moreover, every stable conjugate of a cubic torus is conjugate to it. Therefore, this orbital integral is stable. From [28], the Fourier transform of a stable distribution is stable. Thus, $\theta_\pi|_{G_{0+}}$ is stable. If $z \in Z$, it is clear that $\theta_\pi(\gamma z) = \theta_\pi(\gamma' z)$ if and only if $\theta_\pi(\gamma) = \theta_\pi(\gamma')$. Thus, $\theta_\pi|_{ZG_{0+}}$ is stable.

It follows that (4.2.1) holds if $\gamma \in ZT_{0+}$. Therefore, suppose that $\gamma \notin ZT_{0+}$. We will show that θ_π vanishes at all stable conjugates of γ (including γ itself), thus establishing (4.2.1). Let γ'' be a stable conjugate of γ . If no conjugate of γ'' is contained in G_y , then $\theta_\pi(\gamma'') = 0$ from Proposition 7.1. So assume $\gamma'' \in G_y$. It follows easily from our assumptions on γ that the characteristic polynomial of the image $\bar{\gamma}''$ of γ'' in \mathbf{G}_y is reducible over k_E and that its roots are not all the same. But then the semisimple part of $\bar{\gamma}''$ is not contained in a cubic torus of \mathbf{G}_y so, by [27, 6.9], it follows that $\theta_{\bar{\sigma}}(\bar{\gamma}'') = 0$. Thus $\theta_\pi(\gamma'') = 0$ by Proposition 7.1.

Conversely, suppose that π is not of the form given in the statement of the proposition. By the classification in §2.7, it follows that π is of type (2.7SC-2), (2.7SC-3), or (2.7SC-4).

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in C \subset G_y$ have regular image $\bar{\gamma}$ in \mathbf{G}_y . Let $\gamma' \in G_z$ be the conjugate of γ by ν (see §2.3). Then $\gamma \in G^{\text{reg}}$, and γ lies in a unique maximal parahoric by Lemma 2.1, namely G_y . Also, the image $\bar{\gamma}'$ of γ' in \mathbf{G}_z is regular elliptic, so that γ' is not contained in any parahoric other than G_z . We note that γ and γ' are stably conjugate elements of G that are not conjugate in G .

If π is of type (2.7SC-2) or (2.7SC-3), then π is compactly induced from the inflation σ to G_y of a non-cubic cuspidal representation $\bar{\sigma}$ of \mathbf{G}_y . Thus $\theta_\pi(\gamma') = 0$ by Proposition 7.1 since γ' is not contained in any conjugate of G_y . On the other hand, since the only conjugate of G_y containing γ is G_y , $\theta_\pi(\gamma) = \theta_\sigma(\bar{\gamma})$ by Proposition 7.1.

Suppose that π is of type (2.7SC-3), i.e., $\bar{\sigma} = \tau \cdot (\eta \circ \det)$ where τ is the cuspidal unipotent representation of \mathbf{G}_y and $\eta \in \text{Hom}(k_E^1, \mathbb{C}^\times)$. Then

$$\theta_{\bar{\sigma}}(\bar{\gamma}) = 2(\eta \otimes \eta \otimes \eta)(\bar{\gamma}) \neq 0$$

by [15, p. 31]. On the other hand, suppose that π is of type (2.7SC-2), i.e., the character of $\bar{\sigma}$ is $-R_C^{\mathbf{G}_y} \varphi$, where $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ is a character of \mathbf{C} in general position. Then, by [27, 6.9],

$$\theta_{\bar{\sigma}}(\bar{\gamma}) = - \sum_{w \in W_{k_F}(\mathbb{C}, \mathbf{G}_y)} w(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)(\bar{\gamma}).$$

An easy application of the character theory of abelian groups shows that there is some γ of above type for which this sum does not vanish. Thus if π is of type (2.7SC-2) or (2.7SC-3), then π is not stable. Similarly, if π is compactly induced from G_z (i.e., π is of type (2.7SC-4)), then one can find stably conjugate γ and γ' such that $\theta_\pi(\gamma) = 0$, but $\theta_\pi(\gamma') \neq 0$. Hence π is again not stable. \square

4.3. Non-singleton L -packets containing supercuspidals. Let $\gamma \in C$ be a regular element of G whose image $\bar{\gamma}$ in \mathbf{C} is a regular element of \mathbf{G}_y . Let $\gamma' \in C'$ be the conjugate of γ by ν . Since $\gamma \in G$ and $\bar{\gamma} \in \mathbf{G}$ are regular elliptic, Lemma 2.1 implies that γ lies in a unique maximal parahoric subgroup, namely G_y . Similarly, γ' is not contained in any parahoric subgroup other than G_z . The following lemma then follows easily from Proposition 7.1, [27, 6.9], and [15, p. 31].

Lemma 4.3. *Let π be a supercuspidal representation of G of depth zero. Then, in the notation of §2.7,*

$$\theta_\pi(\gamma) = \begin{cases} - \sum_{w \in W_{k_F}(\underline{G}, \underline{G}_y)} w(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)(\bar{\gamma}) & \text{if } \pi \text{ is of type (2.7SC-2),} \\ 2(\eta \otimes \eta \otimes \eta)(\bar{\gamma}) & \text{if } \pi \text{ is of type (2.7SC-3),} \\ 0 & \text{if } \pi \text{ is of type (2.7SC-4),} \end{cases}$$

$$\theta_\pi(\gamma') = \begin{cases} 0 & \text{if } \pi \text{ is of type (2.7SC-2),} \\ 0 & \text{if } \pi \text{ is of type (2.7SC-3),} \\ - \sum_{w \in W_{k_F}(\underline{G}', \underline{G}_z)} w(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)(\bar{\gamma}) & \text{if } \pi \text{ is of type (2.7SC-4).} \end{cases}$$

There are two types of non-singleton L -packets containing a supercuspidal representation of G of depth zero as discussed in §2.7; namely, the non-supercuspidal L -packets of size two and the supercuspidal L -packets of size four. An L -packet Π of the former type consists of the unique square-integrable constituent $\pi^2 = \pi^2(\lambda)$ (see §2.7) of a reducible principal series of type (2.7PS-2) together with a corresponding supercuspidal representation $\pi^s = \pi^s(\lambda)$. Here λ is a depth-zero character of M such that $\lambda_1|_{F^\times} = \omega_{E/F} \cdot |\cdot|_F^{\pm 1}$. Recall the characters $\lambda_1 \in \text{Hom}(E^\times, \mathbb{C}^\times)$ and $\lambda_2 \in \text{Hom}(E^1, \mathbb{C}^\times)$ determined by λ according to (2.7.1). Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ denote the associated characters of k_E^\times and k_E^1 , respectively.

Proposition 4.4. *Let λ be a depth-zero character of M such that $\lambda_1|_{F^\times} = \omega_{E/F} \cdot |\cdot|_F^{\pm 1}$. Let $\bar{\lambda}'_1$ denote the character of k_E^1 such that $\bar{\lambda}'_1 \circ \mathcal{N} = \bar{\lambda}_1$.*

- (i) *If λ_1 is trivial on \mathcal{O}_E^\times , then $\pi^s(\lambda) = \text{ind}_{G_y}^G \sigma$, where σ is the inflation to G_y of the representation $\tau \cdot (\bar{\lambda}_2 \circ \det)$ of G_y .*
- (ii) *If λ_1 is nontrivial on \mathcal{O}_E^\times , then $\pi^s(\lambda) = \text{ind}_{G_z}^G \sigma$, where σ is the inflation to G_z of the representation of G_z whose character is*

$$-R_{G_z}^G(\bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2).$$

Proof. We determine the supercuspidal representation $\pi^s = \pi^s(\lambda)$ by computing its character at certain regular elliptic elements of G . Recall that the irreducible constituents of $\text{ind}_B^G \lambda$ are $\pi^2(\lambda)$ and a non-tempered representation $\pi^n(\lambda)$ and that the set $\Pi' = \{\pi^s(\lambda), \pi^n(\lambda)\}$ is an A -packet of G . Then Π' is the endoscopic lift from H to G of the character

$$(4.3.1) \quad \xi = (\mu \lambda_2 \circ \det_{U(1,1)}) \otimes \lambda_2$$

of H , where $\mu \circ \mathcal{N} = \lambda_1| \cdot |_E^{\mp 1/2} \omega_{E'/E}$ [23, §12.2, §13.1], and E' is an unramified quadratic extension of E . Let ω be the central character of the elements of Π and let $f \in C(G, \omega)$. By [23, Thm. 13.1.1, Prop. 13.1.2],

$$\theta_{\pi^n}(f) + \theta_{\pi^s}(f) = \theta_\xi(f^H),$$

where $f \mapsto f^H$ is the endoscopic transfer from G to H (see [23, §4.3]). Thus

$$(4.3.2) \quad \theta_{\pi^s} = \theta_\xi^G - \theta_{\pi^n},$$

where θ_ξ^G is the distribution on G that arises from θ_ξ via endoscopy. The same equation holds for the functions on G^{reg} that represent these distributions. Let γ be an element of C whose image $\bar{\gamma}$ in \mathbb{C} is regular. Let $\gamma' \in C'$ be the conjugate of

γ by ν and let $\bar{\gamma}'$ be its image in \mathbf{C}' . In order to determine π^s , we will evaluate the right-hand side of (4.3.2) at γ if $\lambda_1|_{\mathcal{O}_E^\times}$ is trivial and at γ' if $\lambda_1|_{\mathcal{O}_E^\times}$ is nontrivial.

First we compute $\theta_\xi^G(\gamma)$ and $\theta_\xi^G(\gamma')$. By [23, Lemma 12.5.1] and the particular form of γ ,

$$(4.3.3) \quad \theta_\xi^G(\gamma) = \sum_{w \in W_F(\underline{C}, \underline{H}) \setminus W_F(\underline{C}, \underline{G})} \kappa(c_w) \xi({}^w \gamma),$$

where c_w is the class in

$$\mathcal{D}(C/F) := \text{Ker}\{H^1(F, \underline{C}) \rightarrow H^1(F, \underline{G})\}$$

represented by the cocycle $\{s(w)w^{-1}\}$ ($s \in \text{Gal}(\bar{F}/F)$) and κ is the element of the dual of $\mathcal{D}(C/F)$ corresponding to the endoscopic group H . Since $W_F(\underline{C}, \underline{G}) = W(C, G)$, $\kappa(c_w) = 1$ for all $w \in W_F(\underline{C}, \underline{H}) \setminus W_F(\underline{C}, \underline{G})$. Since $W_F(\underline{C}, \underline{G}) \cong S_3$ while $|W_F(\underline{C}, \underline{H})| = 2$, we obtain

$$\theta_\xi^G(\gamma) = \xi((\gamma_1, \gamma_2, \gamma_3)) + \xi((\gamma_3, \gamma_1, \gamma_2)) + \xi((\gamma_2, \gamma_3, \gamma_1)).$$

Evaluating this when $\lambda_1|_{\mathcal{O}_E^\times}$ is trivial and using (4.3.1), we get

$$(4.3.4) \quad 3(\lambda_2 \otimes \lambda_2 \otimes \lambda_2)(\gamma) = 3(\bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}_2)(\bar{\gamma}).$$

As in the preceding paragraph,

$$\theta_\xi^G(\gamma') = \sum_{w \in W_F(\underline{C}', \underline{H}) \setminus W_F(\underline{C}', \underline{G})} \kappa(c_w) \xi({}^w \gamma'),$$

where c_w is now the class in $\mathcal{D}(C'/F)$ represented by $\{s(w)w^{-1}\}$. In this case $W_F(\underline{C}', \underline{G}) \cong S_3$ and $|W_F(\underline{C}', \underline{H})| = 2$. Then $\kappa(c_1) = 1$, and an easy calculation shows that if w represents a nontrivial coset in $W_F(\underline{C}', \underline{H}) \setminus W_F(\underline{C}', \underline{G})$, then $\kappa(c_w) = -1$. Thus

$$\theta_\xi^G(\gamma') = \xi((\gamma_1, \gamma_2, \gamma_3)) - \xi((\gamma_3, \gamma_1, \gamma_2)) - \xi((\gamma_2, \gamma_3, \gamma_1)).$$

We evaluate this when $\lambda_1|_{\mathcal{O}_E^\times}$ is nontrivial. Using (4.3.1), we obtain

$$(4.3.5) \quad (\bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2)(\bar{\gamma}) - (\bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2)(\bar{\gamma}) - (\bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}_2)(\bar{\gamma}).$$

It remains to evaluate θ_{π^n} at γ and γ' . Since γ and $\bar{\gamma}$ are regular elliptic and $\gamma \in G_y$, y is the unique fixed point of γ in \mathcal{B} by Lemma 2.1. Then [24, Lemma III.4.10, Theorem III.4.16] implies that

$$(4.3.6) \quad \theta_{\pi^n}(\gamma) = \text{trace}(\gamma | (\pi^n)^{G_{y+}}).$$

The analogous formula holds for γ' and z . Hence we must determine $(\pi^n)^{G_{y+}}$ and $(\pi^n)^{G_{z+}}$.

Recall that $\pi^n(\lambda)$ and $\pi^2(\lambda)$ are the irreducible constituents of $\text{ind}_B^G \lambda$. Let $\bar{\lambda}$ be the character of \mathbf{M} determined by λ . Since $G = G_y B$, we have that for any $x \in \mathcal{F}$,

$$\text{Res}_{G_y} \text{ind}_B^G \lambda = \text{ind}_{B \cap G_y}^{G_y} \lambda = \text{ind}_{G_x}^{G_y} \text{ind}_{B \cap G_y}^{G_x} \lambda,$$

which contains $\rho_y := \text{ind}_{G_x}^{G_y} \lambda$, the inflation to G_y of the representation $\bar{\rho}_y := \text{ind}_{B_y}^{G_y} \bar{\lambda}$. Since ρ_y is trivial on G_{y+} , this implies that the space of G_{y+} -fixed vectors in $\text{ind}_B^G \lambda$

contains ρ_y . Moreover, by Mackey's theorem and Frobenius reciprocity,

$$\begin{aligned} \mathrm{Hom}_{G_{y+}}(\mathbf{1}, \mathrm{Res}_{G_{y+}} \mathrm{ind}_B^G \lambda) &= \mathrm{Hom}_{G_{y+}}\left(\mathbf{1}, \bigoplus_{g \in G_{y+} \backslash G/B} \mathrm{ind}_{gB \cap G_{y+}}^{G_{y+}} g\lambda\right) \\ &= \bigoplus_{g \in G_{y+} \backslash G/B} \mathrm{Hom}_{gB \cap G_{y+}}(\mathbf{1}, g\lambda) \\ &= \bigoplus_{g \in G_{y+} \backslash G/B} \mathrm{Hom}_{gB \cap G_{y+}}(\mathbf{1}, \mathbf{1}). \end{aligned}$$

The dimension of this space is $|G_{y+} \backslash G/B|$, which (since $G = G_y B$) is equal to

$$|G_{y+} \backslash G_y / (B \cap G_y)| = |G_y / B_y| = \dim \rho_y.$$

Hence the space of G_{y+} -fixed vectors in $\mathrm{ind}_B^G \lambda$ is isomorphic to ρ_y .

Since the vertex z is special, the Iwasawa decomposition $G = G_z \overline{B}$ holds, where \overline{B} is the Borel subgroup opposite B with respect to M . Then an argument similar to that in the preceding paragraph shows that, as a representation of G_z , the space of G_{z+} -fixed vectors in $\mathrm{ind}_B^G \lambda$ is isomorphic to $\bar{\rho}_z = \mathrm{ind}_{B_z}^{G_z} \bar{\lambda}$.

Now let v equal y if $\lambda_1|_{\mathcal{O}_E^\times}$ is trivial or z if $\lambda_1|_{\mathcal{O}_E^\times}$ is nontrivial. Let π be either π^2 or π^n . By [21, Thm. 5.2], for $x \in \mathcal{F}$, $(G_x, \lambda|_{M_0})$ is a K -type contained in π (where we have identified G_x/G_{x+} and M_0/M_{0+}). Thus, as a representation of B_v , $\pi^{G_{x+}}$ contains the character $\bar{\lambda}$ of B_v . By Frobenius reciprocity, $\pi^{G_{v+}}$ contains a subrepresentation of $\bar{\rho}_v$. Since $\bar{\lambda}$ extends to a character of G_v , $\bar{\rho}_v$ is reducible with two irreducible constituents. Replacing λ by a Weyl conjugate if necessary, we may assume that π^2 is a subrepresentation of $\mathrm{ind}_B^G \lambda$, so that we have the exact sequence

$$0 \longrightarrow \pi^2 \longrightarrow \mathrm{ind}_B^G \lambda \longrightarrow \pi^n \longrightarrow 0.$$

Taking G_{v+} -fixed vectors, we obtain the exact sequence

$$0 \longrightarrow (\pi^2)^{G_{v+}} \longrightarrow \bar{\rho}_v \longrightarrow (\pi^n)^{G_{v+}} \longrightarrow 0$$

of representations of G_v . It follows that as a representation of G_v , $\pi^{G_{v+}}$ is an irreducible constituent of $\bar{\rho}_v$.

According to §2.5, the irreducible constituents of $\bar{\rho}_v$ are a one-dimensional representation ψ and the representation $\mathrm{St}_{G_v}(\psi)$. Here

$$\psi = (\bar{\lambda}'_1 \circ \det_{U(1,1)} \circ p_v) \cdot (\bar{\lambda}_2 \circ \det_{\underline{G}_v}),$$

where $p_v : \underline{G}_v \longrightarrow U(1, 1)$ is trivial if $v = y$ or the projection onto the $U(1, 1)$ factor of $\underline{G}_z \cong U(1, 1) \times U(1)$ if $v = z$. Suppose that $(\pi^2)^{G_{v+}} \cong \psi$. Then G_v acts via the character ψ on any nonzero vector $u \in (\pi^2)^{G_{v+}}$. Let $(\pi^2)^\vee$ be the contragredient representation of π^2 . Then G_v acts via ψ^{-1} on any nonzero vector $u' \in ((\pi^2)^\vee)^{G_{v+}}$. An easy computation shows that the matrix coefficient $c_{u, u'}$ is not square-integrable. It follows that $u \notin \pi^2$ and hence that $(\pi^n)^{G_{v+}} \cong \psi$. Thus, if $\lambda_1|_{\mathcal{O}_E^\times}$ is trivial, then from (4.3.6),

$$\theta_{\pi^n}(\gamma) = \mathrm{trace}(\gamma|(\pi^n)^{G_{y+}}) = \psi(\bar{\gamma}) = (\bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}_2)(\bar{\gamma}).$$

On the other hand, if $\lambda_1|_{\mathcal{O}_E^\times}$ is nontrivial, then from (4.3.6),

$$\theta_{\pi^n}(\gamma') = \mathrm{trace}(\gamma'|(\pi^n)^{G_{z+}}) = \psi(\bar{\gamma}') = (\bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2)(\bar{\gamma}).$$

Combining these calculations with (4.3.4), (4.3.5) and (4.3.2), we find that if $\lambda_1|_{\mathcal{O}_E^\times}$ is trivial,

$$(4.3.7) \quad \theta_{\pi^s}(\gamma) = 2(\bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}_2)(\bar{\gamma}),$$

while if $\lambda_1|_{\mathcal{O}_E^\times}$ is nontrivial

$$\theta_{\pi^s}(\gamma) = -(\bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2)(\bar{\gamma}) - (\bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}_2)(\bar{\gamma}).$$

Suppose that $\lambda_1|_{\mathcal{O}_E^\times}$ is trivial. Since π^s is a depth-zero supercuspidal representation, Lemma 4.3 implies that $\theta_{\pi^s}(\gamma)$ is equal to the evaluation at $\bar{\gamma}$ of a linear combination μ of characters of \mathbf{C} depending only on π^s . Letting γ vary over all elements of C that are regular in G and that have regular image $\bar{\gamma}$ in \mathbf{G}_y , we obtain from (4.3.7) that $\mu = 2(\bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}_2)$ on the set of regular elements of \mathbf{C} . By Cor. 2.6, it must be the case that $\mu = 2(\bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}_2)$. By the linear independence of characters of \mathbf{C} , μ must have the character $\bar{\lambda}_2 \otimes \bar{\lambda}_2 \otimes \bar{\lambda}_2$ as a summand. Hence, by Lemma 4.3, π^s must be equivalent to $\text{ind}_{G_y}^G \sigma$, where σ is the inflation to G_y of $\tau \cdot (\bar{\lambda}_2 \circ \det)$. This proves (i). A similar argument with γ' replacing γ proves (ii). \square

We now determine the L -packets of G of size 4. Fix distinct characters χ_1, χ_2 , and χ_3 of k_E^1 . Let χ be the character $\chi_1 \otimes \chi_2 \otimes \chi_3$ of $k_E^1 \times k_E^1 \times k_E^1$. Define regular characters $\chi^{(1)}, \chi^{(2)}$, and $\chi^{(3)}$ of \mathbf{C}' by

$$\begin{aligned} \chi^{(1)} &= \chi \\ \chi^{(2)} &= \chi_2 \otimes \chi_3 \otimes \chi_1 \\ \chi^{(3)} &= \chi_3 \otimes \chi_1 \otimes \chi_2. \end{aligned}$$

Note that each $\chi^{(i)}$ is equal to ${}^w\chi$ for some $w \in W_{k_F}(\underline{\mathbf{C}}', \underline{\mathbf{G}}_z)$. Let σ be the inflation to G_y of the cuspidal representation $\bar{\sigma}$ of \mathbf{G}_y with character $-R_{\mathbf{C}}^{G_y} \chi$. For $i = 1, 2, 3$, let σ_i be the inflation to G_z of the cuspidal representation $\bar{\sigma}_i$ of \mathbf{G}_z with character $-R_{\mathbf{C}'}^{G_z} \chi^{(i)}$. Then $\sigma_1, \sigma_2, \sigma_3$ are distinct by [27, p. 139]. Define $\pi_0 = \text{ind}_{G_y}^G \sigma$ and $\pi_i = \text{ind}_{G_z}^G \sigma_i$ ($i = 1, 2, 3$). By [21], these representations are inequivalent supercuspidals of depth zero. For $v = y$ or z , let σ_v be the inflation to H_v of the cuspidal representation of \mathbf{H}_v with character $-R_{\mathbf{T}}^{H_v} \chi$, where $\mathbf{T} = \mathbf{C}$ if $v = y$, and $\mathbf{T} = \mathbf{C}'$ if $v = z$. Define $\rho_v = \text{ind}_{H_v}^H \sigma_v$. Then ρ_y and ρ_z are inequivalent but conjugate by an element of $\text{PGL}_2(F) \times \{1\}$, and hence $\{\rho_y, \rho_z\}$ is an L -packet for H .

Proposition 4.5. *The set $\{\pi_0, \pi_1, \pi_2, \pi_3\}$ is an L -packet for G and is the endoscopic transfer of $\{\rho, \rho'\}$.*

Proof. Let $R = \{\rho_y, \rho_z\}$ and let Π be the transfer of R from H to G . Then Π has size four by [23, Prop. 13.1.2]. Let $\pi'_0, \pi'_1, \pi'_2, \pi'_3$ be the elements of Π . Then the π'_i are supercuspidal by [23, Prop. 13.1.3(b)]. That they have depth zero follows from our assumption (see the Introduction) that the transfer preserves depth. Set $\theta_R = \theta_{\rho_y} + \theta_{\rho_z}$. Let θ_R^G be the endoscopic transfer of θ_R from H to G . It follows from [23, Thm. 13.1.1, Prop. 13.1.3, Lemma 12.7.2] that

$$(4.3.8) \quad \theta_R^G = \theta_{\pi'_0} + \theta_{\pi'_1} - \theta_{\pi'_2} - \theta_{\pi'_3}$$

for some ordering of the π'_i . Let γ and γ' be as in Proposition 4.4. We will compute $\theta_R(\gamma)$ and $\theta_R(\gamma')$ to determine the π'_i .

Let γ^* be either γ or γ' , and correspondingly let T be either C or C' . According to [23, Lemma 12.5.1], using the notation in the proof of Proposition 4.4,

$$\theta_R^G(\gamma^*) = \sum_{w \in W_F(\underline{T}, \underline{H}) \setminus W_F(\underline{T}, \underline{G})} \kappa(c_w) \theta_R(w\gamma^*).$$

As in the proof of Proposition 4.4, if $\gamma^* = \gamma$, then $\kappa(c_w) = 1$ for all $w \in W_F(\underline{C}, \underline{H}) \setminus W_F(\underline{C}, \underline{G})$, while if $\gamma^* = \gamma'$, then $\kappa(c_1) = 1$ and $\kappa(c_w) = -1$ if w represents a nontrivial coset in $W_F(\underline{C}', \underline{H}) \setminus W_F(\underline{C}', \underline{G})$. Since $\gamma^* \in H$ and $\bar{\gamma}^* \in \mathbf{H}$ are regular elliptic, γ^* lies in a unique maximal parahoric subgroup H_v of H by Lemma 2.1 (where $v = y$ if $\gamma^* = \gamma$, and $v = z$ if $\gamma^* = \gamma'$). Let u be either y or z . It follows from Proposition 7.1 and [27, 6.9] that

$$\theta_{\rho_u}(w\gamma^*) = \begin{cases} - \sum_{u \in W_{k_F}(\underline{T}, \underline{H})} {}^{uw}\chi(\bar{\gamma}^*) & \text{if } u = v \\ 0 & \text{if } u \neq v, \end{cases}$$

where we identify $W_{k_F}(\underline{T}, \underline{H})$ with $W_F(\underline{T}, \underline{H}) \subset W_F(\underline{T}, \underline{G})$. Hence

$$(4.3.9) \quad \begin{aligned} \theta_R^G(\gamma) &= - \sum_{w \in W_{k_F}(\underline{C}, \underline{G})} {}^w\chi(\bar{\gamma}), \\ \theta_R^G(\gamma') &= - \sum_{w \in W_{k_F}(\underline{C}, \underline{G})} d_w {}^w\chi(\bar{\gamma}), \end{aligned}$$

where $d_w = 1$ if $w \in W_{k_F}(\underline{C}, \underline{H})$ and $d_w = -1$ otherwise.

As observed in the proof of Proposition 4.5, Lemma 4.3 implies that $\theta_{\pi'_i}(\gamma)$ is equal to the evaluation at $\bar{\gamma}$ of a linear combination μ_i of characters of \mathbf{C} depending only on π'_i . Therefore, evaluating (4.3.8) at all γ of the above type and using (4.3.9), we obtain

$$- \sum_{w \in W_{k_F}(\underline{C}, \underline{G})} {}^w\chi = \mu_0 + \mu_1 - \mu_2 - \mu_3$$

on the set of regular elements of \mathbf{C} . Then, by Cor. 2.6, this equation must hold at all elements of \mathbf{C} . It follows from Lemma 4.3 and the linear independence of characters of \mathbf{C} that, after possibly reordering, π'_0 must be equivalent to π_0 and that the other elements of the L -packet must be induced from G_z . Evaluating (4.3.8) at γ' of the above type and using a similar argument, we obtain that $\pi'_1 \cong \pi_1$ and, up to reordering, $\pi'_i \cong \pi_i$ for $i = 2, 3$. \square

5. EXPLICIT BASE CHANGE FOR G

5.1. Packets consisting of principal series constituents.

Proposition 5.1. *Let $\lambda \in \text{Hom}(M, \mathbb{C}^\times)$.*

- (i) *If $\text{ind}_B^G \lambda$ is irreducible and $\text{ind}_B^{\tilde{G}} \tilde{\lambda}$ is irreducible, then the base change lift of the L -packet $\{\text{ind}_B^G \lambda\}$ is $\text{ind}_B^{\tilde{G}} \tilde{\lambda}$.*
- (ii) *If $\text{ind}_B^G \lambda$ is irreducible but $\text{ind}_B^{\tilde{G}} \tilde{\lambda}$ is reducible, then $\lambda_1|_{F^\times} = |\cdot|_F^{\pm 1}$, and the base change lift of $\{\text{ind}_B^G \lambda\}$ is $\text{ind}_P^{\tilde{G}} \left((\lambda_1 \tilde{\lambda}_2) \cdot |\cdot|_E^{\mp 1/2} \circ \det_{GL(2)} \right) \otimes \tilde{\lambda}_2$.*
- (iii) *If $\lambda_1 = |\cdot|_E^{\pm 1}$, then the lift of the L -packet comprising the one-dimensional constituent $\psi = \lambda_2 \circ \det_{\underline{G}}$ (respectively, the Steinberg constituent $\text{St}_G(\psi)$) of*

- $\text{ind}_B^G \lambda$ is the one-dimensional constituent $\tilde{\psi} = \tilde{\lambda}_2 \circ \det_{\tilde{G}}$ (respectively, the Steinberg constituent $\text{St}_{\tilde{G}}(\tilde{\psi})$) of $\text{ind}_B^{\tilde{G}} \tilde{\lambda}$.
- (iv) If $\lambda_1|_{F^\times}$ is trivial and λ_1 is nontrivial, then the lift of the L -packet $\{\pi_1(\lambda), \pi_2(\lambda)\}$ is $\text{ind}_B^{\tilde{G}} \tilde{\lambda}$.

Proof. Cases (i), (iii), and (iv) follow from [23] (Prop. 4.10.2 and the paragraph before Theorem 13.2.1). To prove case (ii), note that up to the action of the Weyl group, we may assume that λ is positive with respect to B . The paragraph before Theorem 13.2.1 in [23] then implies that the base change lift of $\{\text{ind}_B^G \lambda\}$ is the Langlands quotient of $\text{ind}_B^{\tilde{G}} \tilde{\lambda}$. This quotient is the desired representation. \square

5.2. Stable supercuspidal representations. Suppose π is a depth-zero, stable, supercuspidal representation of G . From Prop. 4.2, $\pi^{G_y^+}$ contains the inflation σ of a cubic cuspidal representation $\bar{\sigma}$ of $\mathbf{G} \cong \mathbf{G}_y$. Then Figure 1 illustrates how to construct representations $\tilde{\pi}$ and $\tilde{\pi}'$ of $\tilde{G}\Gamma$. We can describe base change for π explicitly by showing that $\tilde{\pi}$ and $\tilde{\pi}'$ are equivalent, provided that the extensions from \tilde{G} to $\tilde{G}\Gamma$ and from $\tilde{\mathbf{G}}$ to $\tilde{\mathbf{G}}\Gamma$ are chosen in compatible ways.

Remark 5.2. Recall the Cartan decomposition for \tilde{G} : The diagonal subgroup \tilde{M} determines a root system Φ for \tilde{G} , and the Borel subgroup \tilde{B} determines a positive root system Φ^+ inside Φ . Let \tilde{M}^+ denote the set of all $m \in \tilde{M}$ such that $\alpha(m)$ has positive valuation for all $\alpha \in \Phi^+$. Then

$$\tilde{G} = \bigcup_{m \in \tilde{M}^+} \tilde{G}_y m \tilde{G}_y.$$

Moreover, $m, m' \in \tilde{M}^+$ represent the same double coset if and only if $m' \in m\tilde{M}_0$.

Lemma 5.3. *Every conjugate of $\tilde{Z}\tilde{G}_y\Gamma$ in $\tilde{G}\Gamma$ is of the form ${}^g m(\tilde{Z}\tilde{G}_y\Gamma)$, where $g \in \tilde{G}_y$, $m \in \tilde{M}^+$.*

Proof. The normalizer of $\tilde{Z}\tilde{G}_y\Gamma$ in $\tilde{G}\Gamma$ is $\tilde{Z}\tilde{G}_y\Gamma$ itself. Therefore, the conjugates of $\tilde{Z}\tilde{G}_y\Gamma$ correspond to the cosets in $\tilde{G}\Gamma/\tilde{Z}\tilde{G}_y\Gamma \cong \tilde{G}/\tilde{Z}\tilde{G}_y$. The lemma now follows from the Cartan decomposition. \square

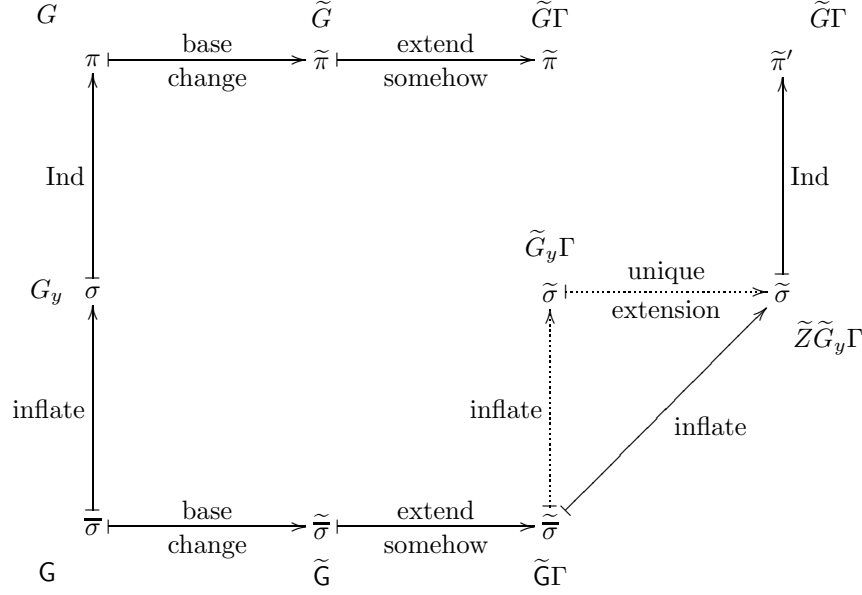
Remark 5.4. Recall that an inner automorphism of \tilde{G} acts on the extended Dynkin diagram either trivially or via a rotation. Thus, if $g \in \tilde{G}$ stabilizes the alcove $\Delta \subset \tilde{\mathcal{B}}^{\text{red}}$ and fixes some point in the closure of Δ not equal to the barycenter of Δ , then g must fix Δ pointwise.

In the next result, we use the fact that $\tilde{\mathbf{G}}_y$ (in addition to being a quotient of \tilde{G}_y) is a quotient of $\tilde{Z}\tilde{G}_y$.

Proposition 5.5. *Let σ be the inflation to G_y of a cubic cuspidal representation $\bar{\sigma}$ of \mathbf{G}_y , and let $\tilde{\sigma}$ be the Shintani lift of $\bar{\sigma}$ from \mathbf{G}_y to $\tilde{\mathbf{G}}_y$. Let $\pi = \text{ind}_{G_y}^G \sigma$. Then the base change lift of the L -packet $\{\pi\}$ is*

$$\text{ind}_{\tilde{Z}\tilde{G}_y}^{\tilde{G}} \tilde{\sigma},$$

where $\tilde{\sigma}$ is the inflation to $\tilde{Z}\tilde{G}_y$ of $\tilde{\sigma}$.

FIGURE 1. Two ways of constructing representations of $\tilde{G}\Gamma$

Proof. Let $\tilde{\pi}$ be the base change lift of $\{\pi\}$ and let $\tilde{\pi}' = \text{ind}_{\tilde{Z}\tilde{G}_y}^{\tilde{G}} \tilde{\sigma}$. Since θ_π is stable by Proposition 4.2, $\theta_{\tilde{\pi}, \varepsilon}$ is a stable ε -class function on $\tilde{G}^{\varepsilon\text{-reg}}$ according to [23, §12.5]. By [21, Prop. 6.8], $\tilde{\pi}'$ is a supercuspidal representation of \tilde{G} of depth zero. Also,

$$\varepsilon \tilde{\pi}' \cong \text{ind}_{\tilde{Z}\tilde{G}_y}^{\tilde{G}} \varepsilon \tilde{\sigma} \cong \tilde{\pi}'$$

since $\tilde{\sigma}$ is ε -invariant as it is in the image of the Shintani lift. Since $\tilde{\pi}'$ is ε -invariant, $\tilde{\pi}'$ is the base change lift of a singleton supercuspidal L -packet $\{\pi'\}$ by [23, Prop. 13.2.2]. But then, as in the case of $\tilde{\pi}$, $\theta_{\tilde{\pi}', \varepsilon}$ is a stable ε -class function. Moreover, it is easily seen (under the assumption that π has depth zero) that the central characters of $\tilde{\pi}$ and $\tilde{\pi}'$ are identical. Furthermore, according to [23, §13.2], we may choose $\tilde{\pi}(\varepsilon)$ and $\tilde{\pi}'(\varepsilon)$ so that $\theta_{\tilde{\pi}, \varepsilon} = \theta_\pi \circ \mathcal{N}$ and $\theta_{\tilde{\pi}', \varepsilon} = \theta_{\pi'} \circ \mathcal{N}$ (see §2.1).

If α_1 and α_2 are stable ε -class functions on $\tilde{G}^{\varepsilon\text{-reg}}$ that transform under \tilde{Z} via the same character, then the ε -elliptic inner product of α_1 and α_2 (see [23, §12.5]) is defined by

$$(5.2.1) \quad \langle \alpha_1, \alpha_2 \rangle_\varepsilon = \sum_{T \in \mathcal{C}} |W_F(\underline{T}, \underline{G})|^{-1} \int_{\tilde{Z}\tilde{T}^{\mathcal{N}} \backslash \tilde{T}} D_G(\mathcal{N}(\delta))^2 \alpha_1(\delta) \overline{\alpha_2(\delta)} d\delta,$$

where \mathcal{C} is a set of representatives for the stable conjugacy classes of elliptic Cartan subgroups of G , \tilde{T} is the centralizer of the Cartan subgroup T in \tilde{G} , $\tilde{T}^{\mathcal{N}}$ is the kernel of the norm map on \tilde{T} , and D_G is the discriminant.

By [23, Prop. 12.6.2], in order to prove that $\tilde{\pi} \cong \tilde{\pi}'$, it suffices to show that

$$\langle \theta_{\tilde{\pi}', \varepsilon}, \theta_{\tilde{\pi}, \varepsilon} \rangle_\varepsilon \neq 0.$$

We will verify the non-vanishing of this inner product by showing that the two twisted characters agree on \tilde{T} for each $T \in \mathcal{C}$.

By Proposition 4.2, the stability of π' implies that π' , like π , is induced from the inflation σ' to G_y of a cubic cuspidal representation $\bar{\sigma}'$ of \mathbf{G}_y . Since $\bar{\sigma}$ and $\bar{\sigma}'$ both arise via Deligne-Lusztig induction from a cubic Cartan subgroup of \mathbf{G}_y , these representations agree on unipotent elements of \mathbf{G}_y [27, 6.9]. It follows that θ_π and $\theta_{\pi'}$ agree on G_{0+} . Since π and π' have the same central character, θ_π and $\theta_{\pi'}$ agree on ZG_{0+} .

Let $T \in \mathcal{C}$, $\delta \in \tilde{T} \cap \tilde{G}^{\varepsilon\text{-reg}}$, and $\gamma = \mathcal{N}(\delta)$. If $\gamma \in ZT_{0+}$, then

$$\theta_{\tilde{\pi},\varepsilon}(\delta) = \theta_\pi(\gamma) = \theta_{\pi'}(\gamma) = \theta_{\tilde{\pi}',\varepsilon}(\delta).$$

We may therefore assume that $\gamma \in T \setminus ZT_{0+}$.

If no conjugate of γ is contained in G_y , then both $\theta_{\tilde{\pi},\varepsilon}(\delta) = \theta_\pi(\gamma)$ and $\theta_{\tilde{\pi}',\varepsilon}(\delta) = \theta_{\pi'}(\gamma)$ vanish by Proposition 7.1. We may therefore assume that $\gamma \in G_y$.

We may assume that T is not of type (2.4-0), since such tori are not elliptic.

Suppose that T is of type (2.4-1) or (2.4-2). As in the proof of Proposition 4.2, since $\gamma \in T \setminus ZT_{0+}$, the semisimple part of the image $\bar{\gamma}$ of γ in \mathbf{G}_y is not contained in a cubic torus of \mathbf{G}_y . Therefore, by [27, 6.9], $\theta_{\bar{\sigma}}$ and $\theta_{\bar{\sigma}'}$ vanish on γ . Thus, again we have

$$\theta_{\tilde{\pi},\varepsilon}(\delta) = \theta_\pi(\gamma) = 0 = \theta_{\pi'}(\gamma) = \theta_{\tilde{\pi}',\varepsilon}(\delta)$$

by Proposition 7.1.

Now suppose that T is of type (2.4-3). Then there exist cubic extensions L of E and K of F such that $L = EK$ and $T \cong \text{Ker}(N_{L/K})$. In particular, we may identify T_{0+} with $\text{Ker}(N_{L/K}) \cap (1 + \mathfrak{p}_L)$ and Z with E^1 . We therefore have $ZT_{0+} \cong E^1 [\text{Ker}(N_{L/K}) \cap (1 + \mathfrak{p}_L)]$. If L/E is totally ramified, then $E^1 [\text{Ker}(N_{L/K}) \cap (1 + \mathfrak{p}_L)] = \text{Ker}(N_{L/K})$ so $T \setminus ZT_{0+}$ is empty, and there is nothing to prove in this case. We may hence assume that L/E is unramified. Since T is determined only up to stable conjugacy, we may also assume that T fixes the point y .

Let $\bar{\gamma}$ be the image of γ in the cubic torus $\mathbf{T} \subset \mathbf{G}_y$. Since $\gamma \notin ZT_{0+}$, $\bar{\gamma}$ is not central in \mathbf{G}_y . Thus $\bar{\gamma}$ is regular elliptic, so by Lemma 2.1, γ is contained in a unique parahoric subgroup of G , namely G_y . Thus

$$\theta_{\tilde{\pi},\varepsilon}(\delta) = \theta_\pi(\gamma) = \theta_{\bar{\sigma}}(\bar{\gamma})$$

by Proposition 7.1. It suffices to show that $\theta_{\tilde{\pi}',\varepsilon}(\delta) = \theta_{\bar{\sigma}'}(\bar{\gamma})$.

Extend $\tilde{\pi}'$ to a representation (also denoted $\tilde{\pi}'$) of $\tilde{G}\Gamma$ in a manner compatible with the choice of $\tilde{\pi}'(\varepsilon)$ made in the beginning of the proof. Then

$$\theta_{\tilde{\pi}',\varepsilon}(\delta) = \theta_{\tilde{\pi}'}(\delta\varepsilon).$$

As a representation of $\tilde{G}\Gamma$,

$$\tilde{\pi}' \cong \text{ind}_{\tilde{Z}\tilde{G}_y\Gamma}^{\tilde{G}\Gamma} \tilde{\sigma},$$

where $\tilde{\sigma}$ is extended compatibly from \tilde{G}_y to $\tilde{G}_y\Gamma$. This extension determines an extension of $\tilde{\sigma}$ to $\tilde{G}_y\Gamma$, and we let $\theta_{\tilde{\sigma}}$ be the corresponding twisted character.

By Proposition 7.1, to compute $\theta_{\tilde{\pi}'}(\delta\varepsilon)$, we must determine which conjugates of $\tilde{Z}\tilde{G}_y\Gamma$ contain $\delta\varepsilon$. Since

$$\tilde{T}^{\mathcal{N}}\tilde{T}_0 \cong K^\times \mathcal{O}_L^\times = L^\times \cong \tilde{T},$$

and since $\theta_{\tilde{\pi}',\varepsilon}(\delta)$ only depends on δ modulo $\tilde{T}^{\mathcal{N}}$, we may assume that $\delta \in \tilde{T}_0$. Since T fixes y , $\delta \in \tilde{T}_0 \subset \tilde{G}_y$ so $\delta\varepsilon \in \tilde{G}_y\Gamma$.

Now suppose that $\delta\varepsilon$ is also contained in another conjugate of $\tilde{Z}\tilde{G}_y\Gamma$. By Lemma 5.3, any such conjugate is of the form ${}^{gm}(\tilde{Z}\tilde{G}_y\Gamma)$, where $g \in \tilde{G}_y$ and $m \neq 1$ is in \tilde{M}^+ . If $\delta\varepsilon \in {}^{gm}(\tilde{Z}\tilde{G}_y\Gamma)$, then $({}^{gm})^{-1}(\delta\varepsilon) \in \tilde{Z}\tilde{G}_y\Gamma$ so

$$m^{-1}\delta'\varepsilon(m) \in \tilde{Z}\tilde{G}_y,$$

where $\delta' = g^{-1}\delta\varepsilon(g) \in \tilde{G}_y$. Thus $\delta'\varepsilon(m) \in m\tilde{Z}\tilde{G}_y$ so $\varepsilon(m)$ and mc represent the same double coset in $\tilde{G}_y \backslash \tilde{G} / \tilde{G}_y$ for some $c \in \tilde{Z}$. Since $\varepsilon(m), mc \in \tilde{M}^+$, we have $\varepsilon(m) \in mc\tilde{M}_0$ by Remark 5.2. It follows that δ' is in ${}^m(\tilde{Z}\tilde{G}_y) = \tilde{G}_{my}\tilde{Z}$ as well as \tilde{G}_y .

Let \bar{y} be the image of y in $\tilde{\mathcal{B}}^{\text{red}}$. Then δ' fixes \bar{y} and $m\bar{y}$, hence fixes the line segment $[\bar{y}, m\bar{y}]$ in $\tilde{\mathcal{B}}^{\text{red}}$. Since $\varepsilon(m) \in mc\tilde{M}_0$, we have $\varepsilon(m\bar{y}) = m\bar{y}$ so that $[\bar{y}, m\bar{y}]$ intersects the (open) triangle $\Delta \subset \tilde{\mathcal{B}}^{\text{red}}$ non-trivially. Hence δ' stabilizes Δ . From Remark 5.4, δ' must fix Δ pointwise. Thus δ' is contained in $\tilde{G}_{\tilde{\mathcal{F}}}\tilde{Z}$, where $\tilde{G}_{\tilde{\mathcal{F}}}$ is the standard upper-triangular Iwahori subgroup of \tilde{G} . The image $\bar{\delta}'$ of δ' in $\tilde{\mathcal{G}}_y$ is therefore contained in the Borel subgroup $\tilde{\mathcal{B}}_y$ of upper-triangular matrices in $\tilde{\mathcal{G}}_y$. Since $\tilde{\mathcal{B}}_y$ is ε -invariant, $\bar{\delta}'\varepsilon(\bar{\delta}')$ is also contained in $\tilde{\mathcal{B}}_y$. Hence the eigenvalues of $\bar{\delta}'\varepsilon(\bar{\delta}')$ lie in k_E^\times . But $\bar{\delta}'\varepsilon(\bar{\delta}') = \bar{\mathcal{N}}(\bar{\delta}') = \bar{\mathcal{N}}(\bar{g}^{-1}\bar{\delta}\varepsilon(\bar{g})) = \bar{g}^{-1}\bar{\gamma}\bar{g}$, where \bar{g} is the image of g in $\tilde{\mathcal{G}}_y$, so the eigenvalues of $\bar{\delta}'\varepsilon(\bar{\delta}')$ are the same as those of $\bar{\gamma}$. The eigenvalues of $\bar{\gamma}$, however, lie in $k_L^1 \setminus k_E^1$ since $\bar{\gamma}$ is a regular element of the cubic torus \mathbf{T} . This contradiction shows that $\delta\varepsilon$ is contained in a unique conjugate of $\tilde{Z}\tilde{G}_y\Gamma$, namely $\tilde{Z}\tilde{G}_y\Gamma$ itself.

We therefore have from Proposition 7.1 that

$$\theta_{\tilde{\pi}', \varepsilon}(\delta) = \theta_{\tilde{\pi}'}(\delta\varepsilon) = \theta_{\tilde{\sigma}}(\bar{\delta}\varepsilon) = \theta_{\tilde{\sigma}, \varepsilon}(\bar{\delta}).$$

But $\tilde{\sigma}$ is the Shintani lift of $\bar{\sigma}$ (see [17]) so the last expression is equal to

$$\pm\theta_{\bar{\sigma}}(\bar{\gamma}).$$

(Here the twisted character $\theta_{\tilde{\sigma}, \varepsilon}$ as chosen above is not a priori equal to $\theta_{\bar{\sigma}} \circ \mathcal{N}$ since this choice is not necessarily the one that is compatible with the Shintani lifting. Nevertheless, it is at worst off by a sign by the discussion in 2.1.) At the same time

$$\theta_{\tilde{\pi}', \varepsilon}(\delta) = \theta_{\pi'}(\gamma) = \theta_{\bar{\sigma}'}(\bar{\gamma}),$$

so $\theta_{\bar{\sigma}'}(\bar{\gamma}) = \pm\theta_{\bar{\sigma}}(\bar{\gamma})$. It is easily seen (e.g., from the character table in [15]) that there is no cubic cuspidal representation $\bar{\sigma}'$ of \mathbf{G}_y satisfying $\theta_{\bar{\sigma}'}(\bar{\gamma}) = -\theta_{\bar{\sigma}}(\bar{\gamma})$ for all regular elements $\bar{\gamma}$ of cubic tori. Thus

$$\theta_{\tilde{\pi}', \varepsilon}(\delta) = \theta_{\bar{\sigma}}(\bar{\gamma}),$$

and the theorem follows. \square

5.3. Non-singleton L -packets containing supercuspidals.

Proposition 5.6. *Let λ be a character of M of depth zero such that $\lambda_1|_{F^\times} = \omega_{E/F}| \cdot |\cdot|_F^{\pm 1}$.*

(i) *The base change lift of the L -packet $\{\pi^2(\lambda), \pi^s(\lambda)\}$ is*

$$\text{ind}_{\tilde{P}}^{\tilde{G}} \left(\text{St}_{\tilde{H}} \left((\lambda_1 \tilde{\lambda}_2 | \cdot |_E^{\mp 1/2} \circ \det_{GL(2)}) \otimes \tilde{\lambda}_2 \right) \right).$$

(ii) The base change lift of the A -packet $\{\pi^n(\lambda), \pi^s(\lambda)\}$ is

$$\mathrm{ind}_{\tilde{P}}^{\tilde{G}} \left((\lambda_1 \tilde{\lambda}_2) \cdot |\cdot|_E^{\mp 1/2} \circ \det_{GL(2)} \right) \otimes \tilde{\lambda}_2.$$

Moreover, the above two base change lifts are precisely the irreducible constituents of the principal series representation $\mathrm{ind}_B^{\tilde{G}}(\tilde{\lambda})$.

Note that the proposition has the same content if we restrict the choice of exponent in the hypothesis to be $+1$ (or to be -1).

Proof. This follows from [23, §§12–13]. More precisely, let ξ be the character

$$(\mu\lambda_2 \circ \det_{U(1,1)}) \otimes \lambda_2$$

of H , where

$$\mu \circ \mathcal{N} = \lambda_1 | \cdot |_E^{\mp 1/2} \omega_{E'/E},$$

E' an unramified quadratic extension of E . (Here, we are identifying H with $U(1,1)(F) \times U(1)(F)$.) Let $\rho = \mathrm{St}_H(\xi)$. Then, by [23, Prop. 13.1.3(c)], the L -packet $\{\pi^2(\lambda), \pi^s(\lambda)\}$ on G is the lift of the L -packet $\{\rho\}$ on H . It follows from [23, Prop. 13.2.2 (c)] that the base change lift of $\{\pi^2(\lambda), \pi^s(\lambda)\}$ is $\mathrm{ind}_{\tilde{P}}^{\tilde{G}}(\tilde{\rho}')$, where $\tilde{\rho}'$ is the “primed” base change lift (see [23, §11.4]) of ρ from H to \tilde{H} . But by [23, §12.1],

$$\tilde{\rho}' = \mathrm{St}_{\tilde{H}}(\tilde{\xi}'),$$

where $\tilde{\xi}'$ is the character

$$(\omega_{E'/E}(\mu\lambda_2 \circ \mathcal{N}) \circ \det_{GL(2)}) \otimes \tilde{\lambda}_2 = (\lambda_1 \tilde{\lambda}_2) \cdot |\cdot|_E^{\mp 1/2} \circ \det_{GL(2)} \otimes \tilde{\lambda}_2.$$

This proves (i), and (ii) follows analogously from [23, Prop. 13.1.3(d)].

The final statement follows from the proof of [23, Lemma 12.7.6]. \square

Recall the notation of Proposition 4.5. Let Π be the supercuspidal L -packet $\{\pi_0, \pi_1, \pi_2, \pi_3\}$, and let $R = \{\rho_y, \rho_z\}$ be the L -packet of H that transfers to Π .

Proposition 5.7. *The base change lift of the L -packet $\Pi = \{\pi_0, \pi_1, \pi_2, \pi_3\}$ is $\mathrm{ind}_B^{\tilde{G}}\chi^*$, where χ^* is inflation to $\tilde{M} \cong E^\times \times E^\times \times E^\times$ of*

$$\hat{\chi} = \tilde{\chi}_1 \otimes \tilde{\chi}_2 \otimes \tilde{\chi}_3 \in \mathrm{Hom}(\tilde{M}, \mathbb{C}^\times).$$

Proof. This also follows from [23, §§12–13]. Note that since E/F is unramified, \tilde{M} is a quotient of \tilde{M} , so the definition of χ^* makes sense. Let $\tilde{\rho}'$ be the “primed” base change lift (see [23, §11.4]) of R from H to \tilde{H} . By [23, Prop. 13.2.2(c)], the base change lift of Π is $\mathrm{ind}_{\tilde{P}}^{\tilde{G}}(\tilde{\rho}')$. By [23, §12.1], R is the transfer from C to H of some character φ of C . Let θ_φ^H be the distribution on H that arises from $\theta_\varphi = \varphi$ via endoscopy. Let \tilde{B}' be a Borel subgroup of \tilde{H} containing \tilde{C} . Then by [23, §12.1], $\tilde{\rho}'$ is the representation $\mathrm{ind}_{\tilde{B}'}^{\tilde{H}} \tilde{\varphi}$. Hence the base change lift of Π is

$$\mathrm{ind}_{\tilde{P}}^{\tilde{G}} \mathrm{ind}_{\tilde{B}'}^{\tilde{H}} \tilde{\varphi} = \mathrm{ind}_{\tilde{B}'}^{\tilde{G}} \tilde{\varphi}.$$

We now determine φ .

Since R has depth zero, φ must as well. Also,

$$(5.3.1) \quad \theta_\varphi^H = \pm(\theta_\rho - \theta_{\rho'})$$

by [23, Prop. 11.1.1(b)]. The same equation holds for the functions that represent these distributions. Let γ be an element of C whose image $\bar{\gamma} \in \mathbf{C}$ is regular in \mathbf{H}_y . As computed in the proof of Proposition 4.5,

$$\theta_\rho(\gamma) = - \sum_{w \in W_{k_F}(\underline{\mathbf{C}}, \underline{\mathbf{H}})} {}^w \chi(\bar{\gamma}),$$

while $\theta_{\rho'}(\gamma) = 0$. Hence the evaluation of the right side of (5.3.1) at γ is

$$\pm \sum_{w \in W_{k_F}(\underline{\mathbf{C}}, \underline{\mathbf{H}})} {}^w \chi(\bar{\gamma}).$$

The analogue of (4.3.3) for the transfer from C to H implies that

$$\theta_\varphi^H(\gamma) = \sum_{w \in W_F(\underline{\mathbf{C}}, \underline{\mathbf{H}})} \varphi({}^w \gamma) = \sum_{w \in W_{k_F}(\underline{\mathbf{C}}, \underline{\mathbf{H}})} {}^w \bar{\varphi}(\bar{\gamma}),$$

where $\bar{\varphi}$ is the character of \mathbf{C} determined by φ . Using (5.3.1) and letting γ vary over all elements of the above type, it follows that

$$\sum_{w \in W_{k_F}(\underline{\mathbf{C}}, \underline{\mathbf{H}})} {}^w \bar{\varphi} = \pm \sum_{w \in W_{k_F}(\underline{\mathbf{C}}, \underline{\mathbf{H}})} {}^w \chi$$

on the set of regular elements of \mathbf{C} , hence on all of \mathbf{C} by Cor. 2.6. By the linear independence of characters of \mathbf{C} , it follows that $\bar{\varphi} = {}^w \chi$ for some $w \in W_{k_F}(\underline{\mathbf{C}}, \underline{\mathbf{G}})$. Since $\bar{\varphi}$ is in the image of the base change lifting from C to \tilde{C} , it follows from [23, §12.4] that $\bar{\varphi}$ is trivial on elements of \tilde{C} of the form $(\varpi^a, \varpi^b, \varpi^c)$. Since $\bar{\varphi}$ has depth zero, $\bar{\varphi}$ must be the inflation to \tilde{C} of ${}^w \chi$ for some w . Thus ${}^{w^{-1}} \bar{\varphi}$ is the inflation to \tilde{C} of χ , where w is viewed as an element of $W(\tilde{C}, \tilde{G})$. Moreover,

$$\tilde{\pi} = \text{ind}_{\tilde{B}'}^{\tilde{G}} \bar{\varphi} \cong \text{ind}_{\tilde{B}'}^{\tilde{G}} {}^{w^{-1}} \bar{\varphi}.$$

Finally, note that by conjugating by a suitable element, one can send \tilde{B}' , \tilde{C} , and ${}^{w^{-1}} \bar{\varphi}$ respectively to \tilde{B} , \tilde{M} , and χ^* . The theorem follows. \square

6. COMPATIBILITY OF BASE CHANGE AND K -TYPES

In this section we prove the Main Theorem, as stated in §1. Throughout, Π will denote an L -packet of G and $\tilde{\pi}$ the base change lift of Π .

6.1. Principal series L -packets. As in §5.1, suppose Π consists entirely of constituents of the depth-zero principal series $\text{ind}_B^G \lambda$. Since each element of Π has depth zero, $\text{ind}_B^G \lambda$ and hence λ have depth zero by [21, Theorem 5.2]. It follows from [21] that for any $x \in \mathcal{F}$, $(G_x, \lambda|_{M_0})$ is a K -type of each element of Π , where M is identified with G_x/G_{x+} . Similarly, $(\tilde{G}_x, \tilde{\lambda})|_{\tilde{M}_0}$ is a K -type of $\tilde{\pi} = \text{ind}_P^{\tilde{G}} \tilde{\lambda}$ (see Proposition 5.1), where $\tilde{G}_x/\tilde{G}_{x+}$ is identified with \tilde{M} . Denote by $\bar{\lambda}$ the character of M that inflates to $\lambda|_{M_0}$. Then $\tilde{\lambda}|_{\tilde{M}_0}$ is the inflation to \tilde{M}_0 of the character $\tilde{\lambda}$ of \tilde{M} . As required, this is the Shintani lift of $\bar{\lambda}$ from M to \tilde{M} .

6.2. Singleton supercuspidal L -packets. Now suppose that Π is a singleton supercuspidal L -packet $\{\pi\}$ of depth zero. Then, by Proposition 4.2, π is of the form $\text{ind}_{\tilde{G}_y}^G \sigma$, where σ is the inflation to G_y of a cubic cuspidal representation $\bar{\sigma}$ of \mathbf{G}_y . Then (G_y, σ) is a K -type of π by [21, Prop. 6.2]. Similarly, it follows from Proposition 5.5 and [21, Prop. 6.2] that $(\tilde{G}_y, \tilde{\sigma})$ is a K -type of $\tilde{\pi}$, where $\tilde{\sigma}$ is the inflation to \tilde{G}_y of the Shintani lift $\tilde{\bar{\sigma}}$ of $\bar{\sigma}$ from \mathbf{G}_y to $\tilde{\mathbf{G}}_y$. Hence the theorem holds in this case.

6.3. Supercuspidal L -packets of size four. Recalling the notation of Proposition 4.5, suppose that $\Pi = \{\pi_0, \pi_1, \pi_2, \pi_3\}$ is a depth-zero supercuspidal L -packet. By [21, Prop. 6.2], (G_y, σ) and (G_z, σ_i) are K -types for π_0 and the π_i ($i = 1, 2, 3$), respectively.

According to Proposition 5.7, $\tilde{\pi}$ is the principal series representation $\text{ind}_{\tilde{B}}^{\tilde{G}} \chi^*$, where χ^* is the inflation to $\tilde{M} \cong E^\times \times E^\times \times E^\times$ of

$$\hat{\chi} = \tilde{\chi}_1 \otimes \tilde{\chi}_2 \otimes \tilde{\chi}_3 \in \text{Hom}(\tilde{M}, \mathbb{C}^\times).$$

View $\chi^*|_{\tilde{M}_0}$ as a character of \tilde{G}_x (for any $x \in \mathcal{F}$) under the identification $\tilde{G}_x = \tilde{M}$. Then, by [21, Thm. 5.2], $(\tilde{G}_x, \chi^*|_{\tilde{M}_0})$ is a K -type for $\tilde{\pi}$. Since $\tilde{\pi}$ contains $(\tilde{G}_x, \chi^*|_{\tilde{M}_0})$, it follows that, as a representation of \tilde{B}_y , $\tilde{\pi}^{\tilde{G}_{x+}}$ contains the character $\hat{\chi}$ of \tilde{B}_y . Hence, by Frobenius reciprocity, $\tilde{\pi}^{\tilde{G}_{y+}}$ contains a subrepresentation of $\text{ind}_{\tilde{B}_y}^{\tilde{G}_y} \hat{\chi}$. But $\text{ind}_{\tilde{B}_y}^{\tilde{G}_y} \hat{\chi}$ is irreducible, as $\hat{\chi}$ is in general position, so $\tilde{\pi}^{\tilde{G}_{y+}}$ contains $\text{ind}_{\tilde{B}_y}^{\tilde{G}_y} \hat{\chi}$. This is the Shintani lift of $\bar{\sigma}$ from \mathbf{G}_y to $\tilde{\mathbf{G}}_y$ (see 2.6).

Identify \tilde{G}_z with $\tilde{H} \subset \tilde{G}$. Now, by 2.6, the Shintani lift of $\bar{\sigma}_i$ is $\text{ind}_{\tilde{B}_z}^{\tilde{G}_z} ({}^w \hat{\chi})$ for an appropriate $w \in W_{k_F}(\tilde{M}, \tilde{G})$. The argument in the preceeding paragraph, applied to $\text{ind}_{\tilde{B}}^{\tilde{G}} ({}^w \chi^*) \cong \text{ind}_{\tilde{B}}^{\tilde{G}} \chi^*$ (where we identify $W_{k_F}(\tilde{M}, \tilde{G})$ and $W_F(\tilde{M}, \tilde{G})$), shows that $\tilde{\pi}^{\tilde{G}_{z+}}$ contains $\text{ind}_{\tilde{B}_z}^{\tilde{G}_z} ({}^w \hat{\chi})$.

6.4. L -packets and A -packets of size two. Now suppose Π is an L -packet of the form $\{\pi^2(\lambda), \pi^s(\lambda)\}$ or an A -packet of the form $\{\pi^n(\lambda), \pi^s(\lambda)\}$ for some $\lambda \in \text{Hom}(M, \mathbb{C}^\times)$ of depth zero (see case (2.7PS-2) and (4.1)). Both $\pi^2(\lambda)$ and $\pi^n(\lambda)$ are constituents of the principal series $\text{ind}_B^G \lambda$. It follows from [21] that both $\text{ind}_B^G \lambda$ and λ have depth zero and that for any x in \mathcal{F} , $(G_x, \lambda|_{M_0})$ is a K -type for both of these representations. By Proposition 5.6, $\tilde{\pi}$ is always a constituent of the principal series $\text{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$. Therefore, as above, $(\tilde{G}_x, \tilde{\lambda}|_{\tilde{M}_0})$ is a K -type for $\tilde{\pi}$. But $\tilde{\lambda}|_{\tilde{M}_0}$ is the inflation of $\tilde{\lambda} \in \text{Hom}(\tilde{M}, \mathbb{C}^\times)$, where $\tilde{\lambda} \in \text{Hom}(\tilde{M}, \mathbb{C}^\times)$ is the character that inflates to $\lambda|_{M_0}$. This shows that the theorem is true for $\pi^2(\lambda)$ and $\pi^n(\lambda)$.

It remains to consider $\pi^s(\lambda)$ (both as an element of $\{\pi^s(\lambda), \pi^2(\lambda)\}$ and as one of $\{\pi^s(\lambda), \pi^n(\lambda)\}$). Let λ_1, λ_2 be the respective characters of E^\times, E^1 determined by λ according to (2.7.1). Suppose first that $\lambda_1|_{\mathcal{O}_E^\times}$ is trivial. Then Proposition 4.4 implies that (G_y, σ) is a K -type for $\pi^s(\lambda)$, where σ is the inflation to G_y of $\tau \cdot (\bar{\lambda}_2 \circ \det)$. Here $\bar{\lambda}_2$ is the character of k_E^1 determined by λ_2 , and τ is the cuspidal unipotent representation of \mathbf{G}_y . From §2.6, the Shintani lift of $\tau \cdot (\bar{\lambda}_2 \circ \det_{\mathbf{G}_y})$ from

G_y to \tilde{G}_y is

$$(6.4.1) \quad \tilde{\tau} \cdot (\tilde{\lambda}_2 \circ \det_{\tilde{G}_y}),$$

where $\tilde{\tau}$ is the unipotent representation of \tilde{G}_y that is neither the trivial nor the Steinberg representation. Let $\tilde{\sigma}$ be the inflation of this representation to \tilde{G}_y . Proposition 5.6 states that the base change lift $\tilde{\pi}$ of Π is

$$\text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\rho}',$$

where $\tilde{\rho}'$ is either a one-dimensional representation $\tilde{\xi}'$ of \tilde{H} or $\text{St}_{\tilde{H}}(\tilde{\xi}')$.

Suppose that $\tilde{\rho}' = \tilde{\xi}'$. By Proposition 5.6,

$$\tilde{\xi}' = (\lambda_1 \tilde{\lambda}_2 | \cdot |_E^{\mp 1/2} \circ \det_{\text{GL}(2)}) \otimes \tilde{\lambda}_2.$$

Using Mackey's theorem and Frobenius reciprocity, we have

$$(6.4.2) \quad \begin{aligned} \text{Hom}_{\tilde{G}_y}(\tilde{\sigma}, \text{Res}_{\tilde{G}_y} \text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\xi}') &= \text{Hom}_{\tilde{G}_y}(\tilde{\sigma}, \text{ind}_{\tilde{P} \cap \tilde{G}_y}^{\tilde{G}_y} \tilde{\xi}') \\ &= \text{Hom}_{\tilde{P} \cap \tilde{G}_y}(\tilde{\sigma}, \tilde{\xi}') \\ &= \text{Hom}_{\tilde{P} \cap \tilde{G}_y}(\tilde{\sigma} \cdot \tilde{\xi}'^{-1}, \mathbf{1}), \end{aligned}$$

where we interpret $\tilde{\sigma} \cdot \tilde{\xi}'^{-1}$ as the product of the restriction of each factor to $\tilde{P} \cap \tilde{G}_y$. Identify \tilde{G}_y with \tilde{G} . Since $\lambda_1|_{\mathcal{O}_E^\times}$ is trivial, $\tilde{\xi}'|_{\tilde{P} \cap \tilde{G}_y}$ is the inflation to $\tilde{P} \cap \tilde{G}_y$ of the character $\tilde{\lambda}_2 \circ \det_{\tilde{H}}$ of \tilde{H} . It follows that $\tilde{\sigma} \cdot \tilde{\xi}'^{-1}$ is the restriction to $\tilde{P} \cap \tilde{G}_y$ of the inflation to \tilde{G}_y of $\tilde{\tau}$. Since both $\tilde{\sigma} \cdot \tilde{\xi}'^{-1}$ and $\mathbf{1}$ are trivial on \tilde{G}_{y+} , (6.4.2) can be identified with

$$\text{Hom}_{\tilde{P}}(\tilde{\tau}, \mathbf{1}),$$

where \tilde{P} is the parabolic subgroup of \tilde{G}_y whose inverse image in \tilde{G}_y contains $\tilde{P} \cap \tilde{G}_y$. By Frobenius reciprocity,

$$\text{Hom}_{\tilde{P}}(\tilde{\tau}, \mathbf{1}) = \text{Hom}_{\tilde{G}_y}(\tilde{\tau}, \text{ind}_{\tilde{P}}^{\tilde{G}_y} \mathbf{1}).$$

It is easily seen that $\text{ind}_{\tilde{P}}^{\tilde{G}_y} \mathbf{1}$ has two irreducible components: the trivial representation and $\tilde{\tau}$. Hence

$$\dim_{\mathbb{C}} \text{Hom}_{\tilde{G}_y}(\tilde{\sigma}, \text{Res}_{\tilde{G}_y} \text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\xi}') = 1.$$

In particular, as a representation of \tilde{G}_y , $\tilde{\pi}^{\tilde{G}_{y+}}$ must contain $\tilde{\sigma}$, as required.

Now suppose that $\tilde{\rho}' = \text{St}_{\tilde{H}}(\tilde{\xi}')$. By Proposition 5.6, the representations $\text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\rho}'$ and $\text{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\xi}'$ are the irreducible constituents of $\text{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$. For all $x \in \mathcal{F}$,

$$\text{Res}_{\tilde{G}_y} \text{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda} = \text{ind}_{\tilde{B} \cap \tilde{G}_y}^{\tilde{G}_y} \tilde{\lambda} = \text{ind}_{\tilde{G}_x}^{\tilde{G}_y} \text{ind}_{\tilde{B} \cap \tilde{G}_y}^{\tilde{G}_x} \tilde{\lambda},$$

which contains $\text{ind}_{\tilde{G}_x}^{\tilde{G}_y} \tilde{\lambda}$, the inflation to \tilde{G}_y of the representation $\text{ind}_{\tilde{B}_y}^{\tilde{G}_y} \tilde{\lambda}$. (Here $\tilde{\lambda}$ is the character of \mathbf{M} determined by λ .) Moreover, $\text{ind}_{\tilde{B}_y}^{\tilde{G}_y} \tilde{\lambda}$ contains two copies of the representation (6.4.1) since

$$(\text{ind}_{\tilde{B}_y}^{\tilde{G}_y} \tilde{\lambda}) \cdot (\tilde{\lambda}_2^{-1} \circ \det_{\tilde{G}_y}) = \text{ind}_{\tilde{B}_y}^{\tilde{G}_y} \mathbf{1}$$

contains two copies of $\tilde{\tau}$. Therefore,

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\tilde{G}_y}(\tilde{\sigma}, \operatorname{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}) \geq 2.$$

Since

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\tilde{G}_y}(\tilde{\sigma}, \operatorname{Res}_{\tilde{G}_y} \operatorname{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\xi}') = 1,$$

it follows that

$$\operatorname{Hom}_{\tilde{G}_y}(\tilde{\sigma}, \operatorname{Res}_{\tilde{G}_y} \operatorname{ind}_{\tilde{P}}^{\tilde{G}} \tilde{\rho}') \neq 0.$$

Hence, as above, $\tilde{\pi}^{\tilde{G}_{y+}}$ must contain $\tilde{\sigma}$.

On the other hand, suppose that $\lambda_1|_{\mathcal{O}_E^\times}$ is not trivial. Let $\bar{\lambda}'_1$ be the character of k_E^1 determined by $\bar{\lambda}'_1 \circ \mathcal{N} = \bar{\lambda}_1$. Let σ be the inflation to G_z of the cuspidal representation $\bar{\sigma}$ of G_z with character $-R_{\mathbb{C}^z}^{\mathbb{G}_z} \chi$, where

$$\chi = \bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}'_1 \bar{\lambda}_2 \otimes \bar{\lambda}_2.$$

Then (G_z, σ) is a K -type for $\pi^s(\lambda)$. By §2.6, the Shintani lift $\tilde{\sigma}$ of σ from G_z to \tilde{G}_z is $\operatorname{ind}_{\tilde{B}_z}^{\tilde{G}_z} \tilde{\chi}$, where $\tilde{\chi} = \bar{\lambda}_1 \tilde{\lambda}_2 \otimes \bar{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_2$.

Now in both the L -packet and A -packet cases, $\tilde{\pi}$ is a constituent of $\operatorname{ind}_{\tilde{B}}^{\tilde{G}} \tilde{\lambda}$ by the proof of [23, Lemma 12.7.6]. Hence $(\tilde{G}_x, {}^w \tilde{\lambda}|_{\tilde{M}_0})$ is a K -type of $\tilde{\pi}$, where x is any point in \mathcal{F} and $w \in W_F(\tilde{M}, \tilde{G})$. As in the case above where π is supercuspidal of type (2.7SC-4), it follows by Frobenius reciprocity that $\tilde{\pi}^{\tilde{G}_{z+}}$ contains a subrepresentation of $\operatorname{ind}_{\tilde{B}_z}^{\tilde{G}_z} ({}^w \tilde{\lambda})$, since ${}^w \tilde{\lambda}|_{\tilde{M}_0}$ is the inflation of the character ${}^w \tilde{\lambda}$ of \tilde{M} (where we have identified $W_F(\tilde{M}, \tilde{G})$ and $W_{k_F}(\tilde{M}, \tilde{G})$). Using the fact that $\lambda_1|_{\mathcal{O}_F^\times}$ is trivial, one finds that

$$\tilde{\lambda} = \bar{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_2 \otimes \bar{\lambda}_1 \tilde{\lambda}_2$$

so for an appropriate w ,

$${}^w \tilde{\lambda} = \bar{\lambda}_1 \tilde{\lambda}_2 \otimes \bar{\lambda}_1 \tilde{\lambda}_2 \otimes \tilde{\lambda}_2.$$

Thus $\operatorname{ind}_{\tilde{B}_z}^{\tilde{G}_z} ({}^w \tilde{\lambda}) = \tilde{\sigma}$, so $\tilde{\pi}^{\tilde{G}_{y+}}$ contains the irreducible representation $\tilde{\sigma}$.

7. ON INDUCED CHARACTERS OF NONCONNECTED GROUPS

Let G now denote the group of rational points of a reductive group defined over a nonarchimedean local field. In particular, we do not assume that G is connected. We will, however, assume that G is a semidirect product of its connected component G^0 and its component group Γ , and that G has a Γ -invariant special parahoric subgroup. Suppose H is an open subgroup of G that is compact modulo the center of G . Let ρ denote an irreducible, smooth representation of H , and let π denote the compactly induced representation $\operatorname{ind}_H^G \rho$ of G . Let K denote a compact open subgroup of G .

Proposition 7.1. *For $g \in G^{\text{reg}}$,*

$$\theta_\pi(g) = \sum_{a \in K \backslash G/H} \left(\sum_{b \in KaH/H} \theta_\rho(b^{-1}gb) \right),$$

where θ_ρ is extended to G by zero. For each g , all but finitely many terms of the inner sum vanish.

Proof. This is identical to the proof of Theorem A.14 of [5]. \square

REFERENCES

- [1] J. D. Adler, S. DeBacker, Murnaghan-Kirillov theory for supercuspidal representations of tame general linear groups, *J. Reine Angew. Math.* 575 (2004) 1–35.
- [2] A. Badulescu, Correspondance entre GL_n et ses formes intérieures en caractéristique positive, Ph.D. thesis, Université de Paris-Sud (1999).
- [3] F. Bruhat, J. Tits, Groupes réductifs sur un corps local I: Données radicielles valuées, *Publ. Math. IHES* 41 (1972) 5–251.
- [4] F. Bruhat, J. Tits, Groupes réductifs sur un corps local II: Schémas en groupes. Existence d’une donnée radicielle valuée, *Publ. Math. IHES* 60 (1984) 197–376.
- [5] C. J. Bushnell, G. Henniart, Local tame lifting for $GL(N)$. I. Simple characters, *Inst. Hautes Études Sci. Publ. Math.* (1996) 105–233.
- [6] C. J. Bushnell, G. Henniart, Local tame lifting for $GL(n)$. II. Wildly ramified supercuspidals, *Astérisque* 254 (1999).
- [7] C. J. Bushnell, G. Henniart, Explicit unramified base change: $GL(p)$ of a p -adic field, *J. Number Theory* 99 (1) (2003) 74–89.
- [8] C. J. Bushnell, G. Henniart, Local tame lifting for $GL(n)$. IV. Simple characters and base change, *Proc. London Math. Soc.* (3) 87 (2) (2003) 337–362.
- [9] C. J. Bushnell, P. C. Kutzko, Smooth representations of reductive p -adic groups: structure theory via types, *Proc. London Math. Soc.* (3) 77 (3) (1998) 582–634.
- [10] S. DeBacker, Some applications of Bruhat-Tits theory to harmonic analysis on a reductive p -adic group, *Michigan Math. J.* 50 (2) (2002) 241–261.
- [11] P. Deligne, D. Kazhdan, M.-F. Vignéras, Représentations des algèbres centrales simples p -adiques, in: *Representations of reductive groups over a local field*, Hermann, Paris, 1984, pp. 33–117.
- [12] P. Deligne, G. Lusztig, Representations of reductive groups over finite fields, *Ann. of Math.* 103 (1976) 103–161.
- [13] F. Digne, Shintani descent and \mathcal{L} functions on Deligne-lusztig varieties, in: P. Fong (Ed.), *The Arcata Conference on Representations of Finite Groups*, Vol. 47, Part 2 of *Proc. Symp. Pure Math.*, Amer. Math. Soc., Providence, RI, 1987, pp. 61–68.
- [14] F. Digne, Descente de Shintani et restriction des scalaires, *J. London Math. Soc.* 59 (2) (1999) 867–880.
- [15] V. Ennola, On the characters of the finite unitary groups, *Ann. Acad. Sci. Fenn. Ser. A I* 323 (1963).
- [16] A. Gyoja, Liftings of irreducible characters of finite reductive groups, *Osaka J. Math* 16 (1979) 1–30.
- [17] N. Kawanaka, On the irreducible characters of the finite unitary groups, *J. Math. Soc. Japan* 29 (3) (1977) 425–450.
- [18] C. D. Keys, Principal series representations of special unitary groups over local fields, *Compositio Math.* 51 (1) (1984) 115–130.
- [19] J.-L. Kim, I. I. Piatetski-Shapiro, Quadratic base change of θ_{10} , *Israel J. Math.* 123 (2001) 317–340.
- [20] L. Morris, Tamely ramified intertwining algebras, *Invent. Math.* 114 (1993) 1–54.
- [21] A. Moy, G. Prasad, Jacquet functors and unrefined minimal K -types, *Comment. Math. Helv.* 71 (1) (1996) 98–121.
- [22] J. Rogawski, Representations of $GL(n)$ and division algebras over a p -adic field, *Duke Math. J.* 50 (1) (1983) 161–196.
- [23] J. Rogawski, *Automorphic Representations of Unitary Groups in Three Variables*, Vol. 123 of *Annals of Math. Studies*, Princeton University Press, Princeton, New Jersey, 1990.
- [24] P. Schneider, U. Stuhler, Representation theory and sheaves on the Bruhat-Tits building, *Publ. Math. IHES* 85 (1997) 97–191.
- [25] T. Shintani, Two remarks on irreducible characters of finite general linear groups, *J. Math. Soc. Japan* 28 (2) (1976) 396–414.
- [26] A. J. Silberberger, E.-W. Zink, An explicit matching theorem for level zero discrete series of unit groups of p -adic simple algebras, preprint, 2004.
- [27] B. Srinivasan, *Representations of Finite Chevalley Groups*, Vol. 764 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1979.

- [28] J.-L. Waldspurger, Transformation de Fourier et endoscopie, *J. Lie Theory* 10 (1) (2000) 195–206.
- [29] J.-L. Waldspurger, Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés, *Astérisque* 269 (2001).

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